INTERMEDIATE PHYSICS

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INTERMEDIATE PHYSICS

BY

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PART I

FUNDAMENTAL MEASUREMENTS AND THE GENERAL PROPERTIES OF MATTER

LONDON

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PREFACE

In this book an endeavour has been made to cover the syllabuses required in physics for the Intermediate, the Higher School Certificate, and Scholarship Examinations of the various universities. Although the selection of the material which is to appear is in some measure a matter of personal taste, and other teachers may assess the various parts of the subject differently, it is hoped that a study of the following pages will furnish the student with a comprehensive knowledge of the essential principles of elementary physics, and provide him with a useful tool for further work in this and other subjects. It is this latter aspect which accounts for the somewhat numerous references to the applied sciences. In order thus to equip the student and complete the argument as far as space and the mathematical attainments of the student would permit, the author has not hesitated to use the calculus notation and, in one or two instances, the powerful and beautiful methods of the calculus itself. In this way it is hoped that students will acquire, in the earlier stages of their careers, knowledge which is essential if they are properly to appreciate the aims of physics, and moreover, knowledge which must be possessed before work for a degree in physics is attempted. The author firmly believes that such knowledge must be attained at an early stage if the task of the student in mastering the more advanced parts of his subject is not to be too arduous.

In presenting this third edition to his readers, the author has kept in view three chief aims, viz. (i) to explain in greater detail the more elementary parts of the subject, as well as those parts which usually appear difficult on a first acquaintance with them; (ii) to add an account of that portion of physics essential to scholarship candidates and to those who desire to obtain more than a superficial knowledge of the subject; (iii) to endeavour to give definitions and to use equations which are correct dimensionally. Usually, in an elementary exposition of physics, the dimensions of a physical quantity are not considered—a course leading to much trouble in later years. In order to indicate those parts of the book which are generally considered to be rather above Intermediate standard, they have been printed in smaller

type: such portions should certainly be omitted on a first reading.

In Part I there is a general account of the properties of matter where the subject of surface tension has been treated on the basis of the idea of surface energy, i.e. molecular happenings in the liquid itself. The subjects of diffusion, osmosis, and elasticity, have been treated in a somewhat detailed manner. A brief account of the theory of dimensions and examples of its use have been added.

In Part II an elementary exposition of the subject of heat is presented. Here the author has endeavoured to give brief accounts of some of the more modern and accurate methods of obtaining thermodynamical data. In particular, the fundamental principles of continuous-flow calorimetry have been developed, and the method then applied to the determination of latent heats of vaporization and thermal conductivities. In order to maintain uniformity with the rest of the book, the specific heat of a substance has been defined in such a way that its dimensions are, in one system of units, cal. gm.$^{-1}$ deg.$^{-1}$ C. This seems desirable, since the dimensions of all equations appearing in the subject of heat are then correct. The chapter on thermal conductivity remains greatly extended; attention is there directed to the distribution of temperature in bars along which heat flows under different conditions; a brief account of modern guard-ring methods is given as well as an application of this method to liquids. In the chapter on the first law of thermodynamics the historical development is emphasized. Some parts of the chapter on radiation have been rewritten, the development now being more logical: an effort is also made to draw a clear distinction between processes depending only on the emission or absorption of radiant energy, and those in which the processes of radiation, conduction and convection are simultaneously involved. Moreover, the determination of specific heats by the method of cooling is described in the chapter on calorimetry—not as usual, following an account of Newton's law of cooling, for the method is independent of the validity of this law.

Optics forms the subject of Part III, and here an effort has been made to expound the principles of tracing rays through an optical system; it is only by the actual carrying out of such tracings that a thorough acquaintance with the elementary principles of optical instruments may be obtained. In dealing with the subject of magnification, this has been regarded as a numerical quantity, so that any formulæ for magnification only contain positive entities. These are denoted in the usual manner by $|x|$, etc. Students seem to find this method the least difficult of all. The chapter on optical instruments has been rewritten and considerably extended: the

treatment is now up to date. The subjects of interference, diffraction, and polarization are treated in still greater detail. The theoretical part has been made to depend on ideas involving the time of transit between two points rather than on the number of waves.

In Part IV there follows a brief survey of acoustics, where a short account of the modern methods of sound-ranging on land and sea has been given. Here there appears a comprehensive account of methods for determining the velocity of sound in air and in sea-water: a short section on supersonics is included. The treatment of Lissajou's figures is now more complete.

Part V, that section of the book dealing with electricity and magnetism, has been considerably extended. This section begins with an account of electrostatics, in which there is included a chapter on the theory of isotropic dielectrics. Here the idea of 'electric displacement' is developed and a brief account of Debye's work on the dielectric constants of gases follows. Gauss's theorem and its applications are then discussed. Electrostatic instruments are treated quite fully and in an up-to-date manner. A section on magnetism follows: here, as in electrostatics, the term 'strength of field' is generally used in preference to 'intensity of field'. The symbol H now denotes the strength of a magnetic field, so that another symbol, e.g. H_0, must be selected to denote the horizontal component of the earth's magnetic field. In this section on magnetism there is given a short account of an elementary form of the Schuster magnetometer and of instruments used for recording continuously variations in the magnetic elements. In the opening remarks of the first chapter on current electricity, the connexion between electricity produced by friction and voltaic electricity is discussed. Many changes appear in the succeeding chapters where a fairly full account of accurate methods of measuring a current and a resistance has been given. The underlying ideas have then been applied to the determination of small resistances and of small potential differences. More attention has been given to the design and principles of construction of electrical measuring instruments. The chapter on the magnetic properties of iron and steel has again been enlarged: it includes a brief discussion of paramagnetic and diamagnetic substances. The chapter on electromagnetic induction has been thoroughly revised. Here, as in other parts of the book, greater stress has been laid on historical facts, the pioneer work of Faraday being followed step by step. Many new diagrams showing the lines of magnetic induction (i) due to the original field, (ii) due to the induced current, are shown. The last chapter gives an account of modern work concerning the fascinating story of the atom; it has only been

touched upon briefly—just sufficient perhaps to whet a student's appetite for more, but not sufficient to distract him from the more fundamental parts of the subject.

As in the first edition, the treatment is mainly experimental and most of the graphs and numerical examples in the text are taken from actual observation. No attempt has been made, however, to give all the practical details of the experiments which students are expected to try for themselves, except in some of the more difficult exercises. In many instances graphical methods of dealing with experimental observations have been suggested.

In the near future, the author hopes to publish a text-book of practical physics of intermediate standard, and also a collection of examples, both worked and to be worked.

In all parts numerous diagrams will be found. These generally are in the form of a 'section,' and it is hoped that these will help in an understanding of the text and be found suitable for reproduction when occasion arises. Most of the original drawings have been executed from very sketchy material by my brother, Brigadier L. G. Smith, O.B.E., and to him I wish to express my very best thanks. The author also wishes to thank D. Orson Wood, Esq., M.Sc., for the valuable suggestions which he has continued to give. Thanks are also due to numerous correspondents who have pointed out errors of omission as well as of commission; also to Professor Sir Charles V. Boys, F.R.S., Professor A. Ferguson, D.Sc., Professor L. F. Bates, D.Sc., Dr. J. H. Brinkworth, Dr. H. J. T. Ellingham, and J. Nicol, Esq., B.A., B.Sc., who have made suggestions with regard to the earlier editions or who have gladly given advice when consulted. Lastly, the author would like to express his appreciation of the continued help given by his wife (*née* H. F. Taylor), without whose assistance in all matters connected with this edition its publication would have been delayed still further.

ROYAL HOLLOWAY COLLEGE,
ENGLEFIELD GREEN,
SURREY.
April, 1947.

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[The letters L, B, and N are used in the examples to indicate the source of the question. Also, I denotes Intermediate, S.C. denotes School Certificate, etc.]

THE GREEK ALPHABET

Letters.		*English Equivalent.*	*Names.*
Α	α	ă or ā	alpha
Β	β	b	beta
Γ	γ	g	gamma
Δ	δ	d	delta
Ε	ε	ĕ	epsilon
Ζ	ζ	z	zeta
Η	η	ē	eta
Θ	θ	th	theta
Ι	ι	ĭ or ī	iota
Κ	κ (ϰ)	k	kappa
Λ	λ	l	lambda
Μ	μ	m	mu
Ν	ν	n	nu
Ξ	ξ	x	xi
Ο	ο	ŏ	omikron
Π	π	p	pi
Ρ	ϱ	r	rho
Σ	σ, or (final) ς	s	sigma
Τ	τ	t	tau
Υ	υ	ŭ or ū	upsilon
Φ	ϕ	ph	phi
Χ	χ	kh	khi
Ψ	ψ	ps	psi
Ω	ω	ō	omega

CONTENTS

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FUNDAMENTAL MEASUREMENTS AND THE GENERAL PROPERTIES OF MATTER

INTERMEDIATE PHYSICS

PART I

FUNDAMENTAL MEASUREMENTS AND THE GENERAL PROPERTIES OF MATTER

CHAPTER I

THE MEASUREMENT OF LENGTH, ANGLE, TIME AND MASS

Natural Science.—That branch of human knowledge in which the properties of the material world are examined and then discussed is called natural science. Throughout the ages there have always been those who have endeavoured to become better acquainted with the events around them, whilst, until recently, there has always been a majority who have been content to live in the midst of phenomena about which they knew little ; to them science made little or no appeal. They were disposed rather to regard all phenomena as simple and self-explanatory. At the present time, however, such a state of affairs can hardly be conceived, since the advent of wireless and the extensive use of electricity in daily life have made it almost essential for everyone to become acquainted with the elements of science. But even in the centuries which have gone there have always been those who were not satisfied with a cursory view of Nature, so that they sought to discern the nature of things by careful experimental study. The experimentalist is for ever probing the inner secrets of Nature and, in so doing, he becomes more cognizant of the majesty and mystery of the universe around him. It is very probable that a study of Nature was begun soon after the appearance of Man upon this planet, for it is very difficult to imagine even amongst a tribe of uncouth savages an entire lack of interest in the wonders confronting it. To these people, however, every manifestation of Nature's power was a thing of awe and fear, capable only of being changed by prayer and intercession to the gods and demons which were believed to have their habitations in the material things of this world. Gradually, however, succeeding generations, profiting

by the knowledge handed down to them from their ancestors, began to refer various effects to certain fixed causes. They learned to interpret the signs of the heavens, and put their frail barks to sea when they thought that a period of calm was likely to persist. They became acquainted with the footprints of various animals and knew the times when these animals would come for water. Traps were set, and with the flesh of the animals so caught these people were able to provide for the sustenance of their families; the skins of the animals provided them with raiment for their bodies and also enabled them to erect a cover to protect themselves from the fury of the storm. These ancient inhabitants of the earth were really becoming familiar with laws, for they were realizing that certain causes would inevitably be followed by certain effects.

In every generation a few people were able to add a little to the sum of human knowledge, and each new discovery made further progress more rapid, until, during the last few decades, the advancement of scientific knowledge has been as remarkable as it has been beneficial to mankind. No man is able to claim a thorough acquaintance with all the laws and theories of modern science, so that it has been necessary to divide natural science into several branches, physics, chemistry, biology, etc., and the great strides which have been made in all these branches during the last century and this, have made further subdivision imperative in all these sections of natural science. In biology the properties of living matter are investigated. Here, much of the work is at present only of a qualitative nature, for the processes which are at work are very complicated and intricate, necessitating a vast amount of research before the laws governing them can all become known; recent developments in this field have made it very apparent that there are definite laws and that these laws must be obeyed or the penalty paid. In physics and chemistry the properties of inert matter are examined and the investigations now completed are so extensive that many quantitative laws are known, the discovery and formulation of which have been made possible by the availability of exact standards of measurement. These have arisen from the fact that, if real progress is to be made, exact comparisons must be possible. Amongst the instruments of greatest service in the development of modern science are the balance, thermometer, spectroscope, microscope, and that new and powerful tool the X-ray spectrometer—which has extended our knowledge of the structure of atoms in a manner which would otherwise have been almost impossible. Another reason why exact standards of reference have become so necessary at the present time is that industry is always making demands upon the scientist to supply it with

more accurate tools, or standards for checking the articles it manufactures. A mere mention of the aeroplane, or of the thermionic valve, brings home to us at once the truth of the above statements. We shall therefore begin our study of physics with a short discussion of the fundamental units which form the basis upon which modern science has been built. In passing, a brief reference to the difference between the science of to-day and that which flourished at the time of the ancient Greeks may not be inappropriate. Those early philosophers were content to make observations on a few things and proceed at once to develop a theory, and once having framed it they adhered to it most tenaciously. The procedure during the last few centuries has been very different. Scientists had realized that theories were utterly useless unless they could be substantiated by numerous facts, and they therefore set aside the art of making theories and directed all their attention to establishing facts. When these facts had been correlated, theories became possible, although modern scientists have always recognized that it is the facts which are true and that the theories are merely the product of man; hence, like man, they may be here to-day and gone to-morrow.

The Three Fundamental Units.—The statement that the height of St. Paul's Cathedral is 365 ft. conveys two ideas—one is the unit [the foot], while the other states how many times this unit is contained in the height of the object measured. Later on we shall find that all units, e.g. those of speed, force, electric current, pole strength, etc., may each be expressed in terms of three others, viz. length, mass, and time. These are the ***three fundamental units,*** while all others are called ***derived units.*** The fundamental units in scientific work are the centimetre, gram, and second, so that the system of units based on these particular units of length, mass, and time is referred to as the cm.-gm.-sec. [c.g.s.] system. In England and English-speaking countries, another system is nearly always used for domestic and commercial purposes: it is known as the foot-pound-second [f.p.s.] system because its fundamental units are the foot, pound, and second.

The Measure of Length.—In England the unit of length is the foot, which is defined as one-third the distance between the central traverse lines on two gold plugs in a bronze bar called the ***Imperial standard yard*** when this bar is at 62° F. and supported so that it is not bent when comparisons with it are being made. [Weights and Measures Act, 1878.] A longitudinal section of this bar is shown in Fig. 1·1 (*a*), the two gold plugs being shown in black. It will be observed that the upper surfaces of these plugs on which the fiducial lines are engraved are in the median plane of the bar where the errors due to any possible bending are a minimum.

The unit of length in the c.g.s. system is the centimetre which is defined as the one-hundredth part of the metre. This latter was intended to be one-ten-millionth part of the line of longitude passing through Paris and extending from the North Pole to the Equator. Actually this desire was not quite fulfilled, and so, for legal and scientific purposes, the metre is defined as the length at 0° C. between two fixed lines engraved upon the central flat portion of a platinum-iridium bar, a cross-section of which is indicated in

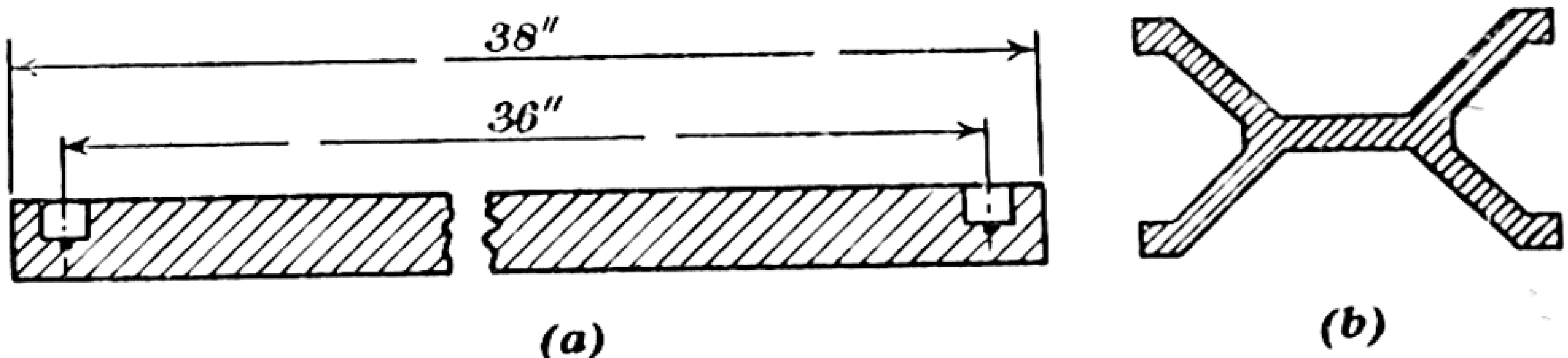


Fig. 1·1.—Standards of Length.

Fig. 1·1 (*b*). This bar is termed the ***International prototype metre.*** Its length in metres at any other temperature is given by

$$l_t = 1 + [(8{\cdot}651t + 0{\cdot}00100t^2) \times 10^{-6}]$$

where l_t is the length at $t°$ C.

The Measure of Mass.—In the British system the pound is the unit of mass; it is defined as the mass of a certain platinum cylinder marked 'P.S. 1844, 1 lb.', and deposited with the Warden of Standards in London. When a copy of this platinum standard is to be made in some other metal it is necessary to allow for the buoyancy of the air so that in recent acts the words 'in vacuo' have been added to define the standard condition of the platinum cylinder.

The metric system adopts as its standard of mass the gram, which is the thousandth part of a mass of platinum-iridium, called the ***International prototype kilogram.*** This latter is very nearly the mass of a cubic decimetre of distilled water at such a temperature that its density is a maximum, viz. 3·98° C. when the pressure on the water is one atmosphere. It is just as necessary to specify the pressure as it is the temperature in the above statement, since the volume of a given mass of water depends upon the external pressure to which it is subjected.

Time.—The choice of a standard of time is more difficult than for the other fundamental units, since, whereas different lengths or masses may be compared with the same respective standard, no standard unit of time is available—time can only be measured by the repetition of a process. The rotation of the earth about its axis is an excellent standard of uniform motion, but it is not quite perfect. Tidal friction increases its period of revolution, while any contraction in its size tends to accelerate its motion. The

other natural clocks which astronomy offers to us are the revolutions of the planets or of the satellites of Jupiter. These are not convenient standards, however, so that they are only used as a last resort to confirm or disprove any variation which may have been suspected in some other standard clock.

The unit of time is the ***mean solar second*** which is the $\frac{1}{86400}$th part of a mean solar day. The solar day is the period which elapses between successive transits of the sun across the meridian at any point on the earth's surface. The duration of a solar day is not a constant magnitude but varies according to the time of the year when it is measured. It is for this reason that the average value of the solar day taken over a twelvemonth is used in defining our unit of time, and this mean value is called the ***mean solar day.*** Astronomers, however, use a different unit of time known as the ***mean sidereal second.*** This is derived from the mean sidereal day which is the average value of the period which elapses between successive transits of one of the fixed stars across a meridian, the average being taken over a period of one year.

In consequence of the earth's orbital motion round the sun, the time interval between two successive transits of the sun across the meridian at any place on the earth is different from that between two successive transits across that meridian of a fixed star. For simplicity, let us assume that the earth's orbit is a circle with the sun, S, Fig. 1·2, at its centre. This circle has a radius $9{\cdot}3 \times 10^6$ miles, and is described in 365 days, 6 hours, 9 min., 9 sec. (solar time)—the length of a so-called ***sidereal year,*** or the time interval between two successive appearances of the sun in the same position relative to the fixed stars. Let E_1 be the position of the earth when a transit of the sun and of a star occur simultaneously. When the earth has made one complete revolution about its axis, i.e. the next transit of the star takes place, it will be

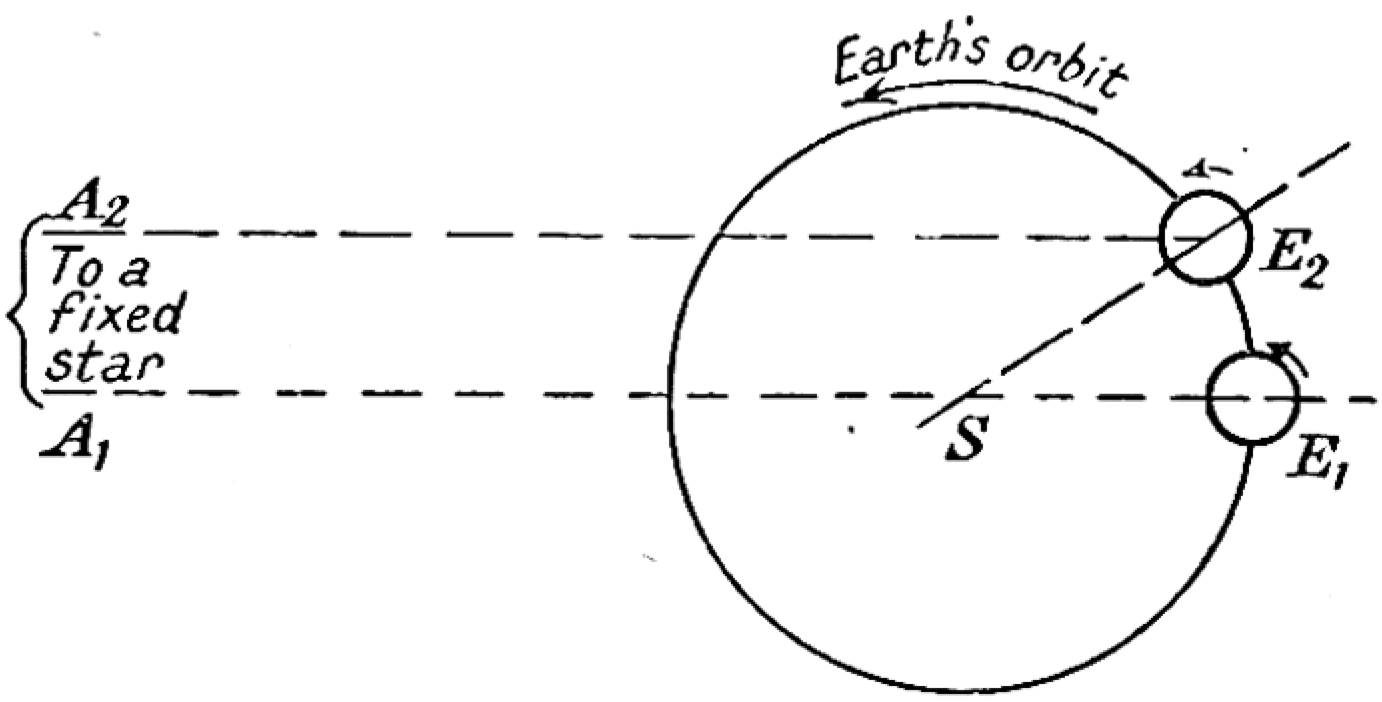


Fig. 1·2.—The Sidereal and Solar Days.

at E_2 but, as the diagram shows, a further rotation through $A_2\widehat{E_2}S$ must occur before the sun crosses the meridian. Hence the solar day is longer than the sidereal day. Actually the mean sidereal day is equal to 23 hours, 56 minutes, 4·09 seconds of mean solar time. [N.B.—If the earth rotated about its axis in the opposite direction, the solar day would be shorter than the sidereal day.]

The Vernier.—When it is desired to determine the distance between two given points it is quite fortuitous if that distance happens to be an exact multiple of the unit of length used; in general there will remain a fraction of a unit for which the relatively coarse divisions on the scale cannot account. This small fraction is determined with the aid of a vernier, the principle of which may be learned from the following: Let AB, Fig. 1·3 (*a*), be a line 9 cm. in length, and let AC and BD be two parallel lines each 10 cm. long. By dividing these two parallel lines into ten equal parts and joining corresponding points by straight lines the line AB is divided into ten equal divisions. This constitutes the vernier scale. Suppose now that the one extremity of a body being measured lies somewhere between the divisions marked 41 and 42 on the main scale. To locate the position of the end of the body more exactly the vernier scale is placed with its zero end in contact with the extremity of the object, when it is observed

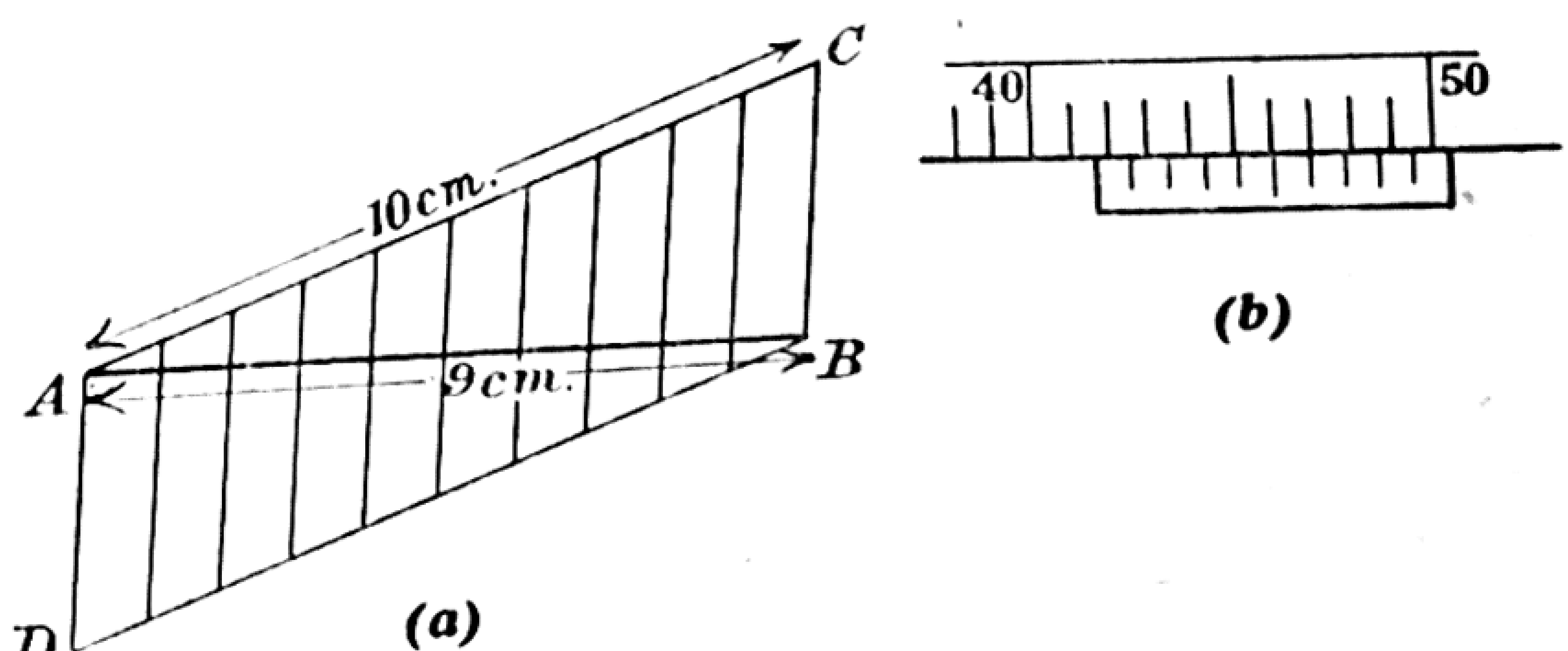


FIG. 1·3.—Principle of a Vernier.

that the sixth division on the vernier scale coincides with a division on the principal scale—cf. Fig. 1·3 (*b*). Since each division on the vernier is one-tenth of nine divisions on the principal scale, i.e. in this particular instance one-tenth of 9 cm., and therefore 0·9 cm., it follows that the difference between one division on the principal scale and one on the vernier is one-tenth of one division on the principal scale, i.e., 0·1 cm. on the vernier constructed above. Hence the difference between six scale divisions and six vernier divisions is $6 \times 0{\cdot}1$ cm., so that the required reading is 41·6 cm.

In actual practice this method of constructing a vernier is always applied to the smallest divisions on the principal scale, i.e. one finds that the vernier is generally 9 mm. long, so that each division on it, if it is divided into tenths, is 0·9 mm. = 0·09 cm. The difference between one division on each of the two scales is then 0·01 cm. When greater accuracy is required, nineteen small divisions on the principal scale are divided into twenty parts so that the difference between one small division on it and one on the vernier is one-twentieth of a small division on the principal scale.

Slide Callipers.—As an actual example of the use of a vernier to determine tenths of a millimetre reference may be made to a pair of slide callipers, Fig. 1·4. Q is the main scale, graduated in cm. and mm., while the vernier V is attached to a movable jaw B. The jaws A and B are perpendicular to the scale Q; the body P whose length is required is inserted between these jaws. When the jaws are closed the zeros of the scale on Q and the vernier V should coincide, whilst when the jaws are open the position of the vernier zero gives, on Q, the perpendicular distance between the jaws. In this particular instance, 10 vernier divisions are equivalent to nine small scale divisions, i.e. to 9 mm., so that each vernier division is equal to 0·9 mm. Now the difference between one small division on the principal scale and one division on the vernier is [0·1 — 0·9 (0·1)] cm. = (0·1 — 0·09) cm. = 0·01 cm. From the figure it is seen that the diameter of the object P is between 7 mm. and 8 mm., and that the seventh division on the vernier coincides with a division on Q; hence the small fractional part which is required is the difference between 7 small scale divisions and 7 vernier divisions. This difference is seven times the difference between 1 principal scale division and 1 vernier division, viz. 7 × 0·01 cm. = 0·07 cm. The length of the object is, therefore, 0·7 + 0·07 = 0·77 cm.

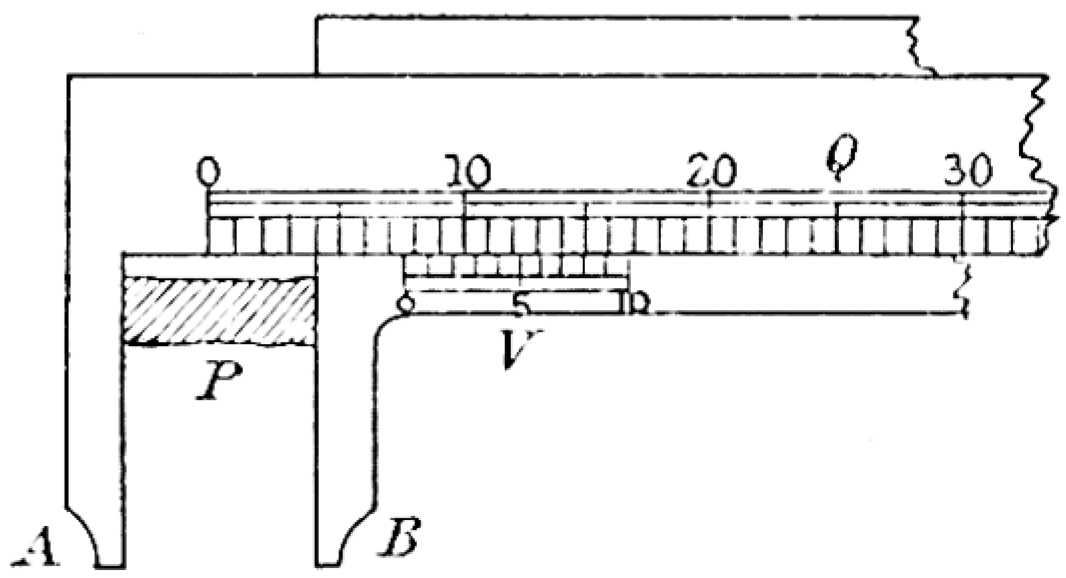


FIG. 1·4.—Slide Callipers.

The Micrometer Screw.—The micrometer screw gauge is another device for measuring small distances accurately. A linear scale in millimetres is engraved parallel to the axis of a cylindrical

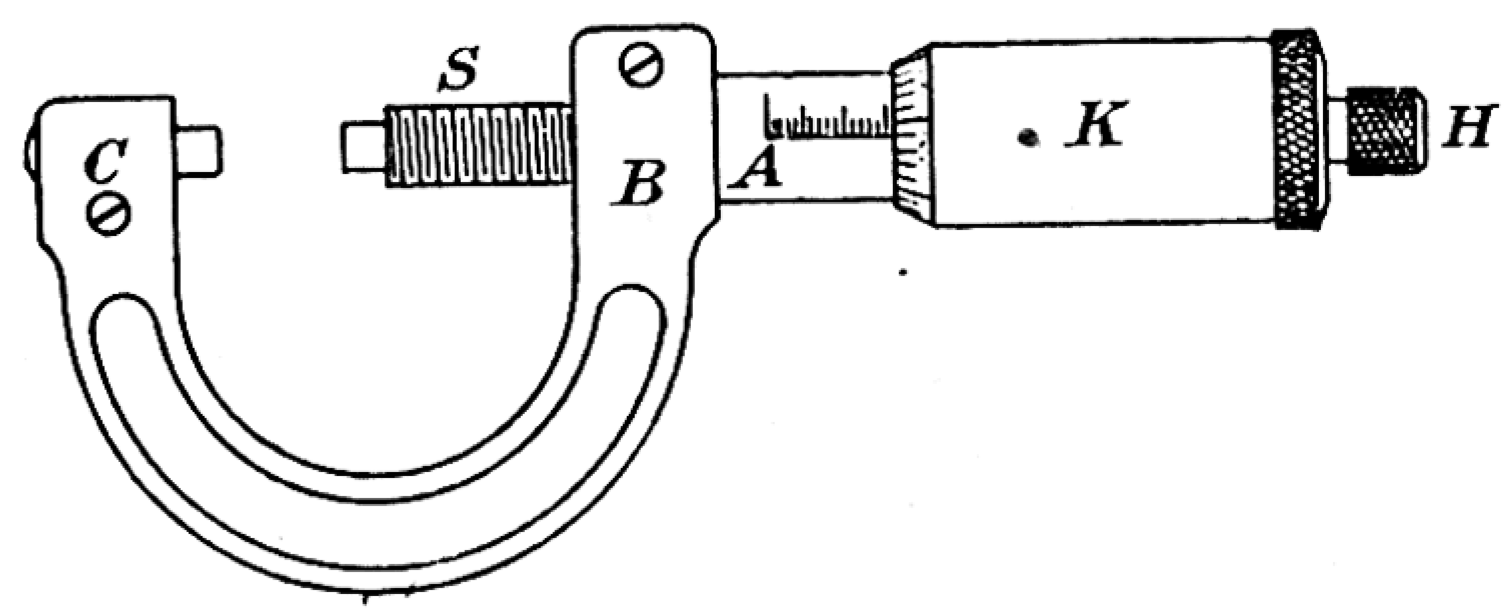
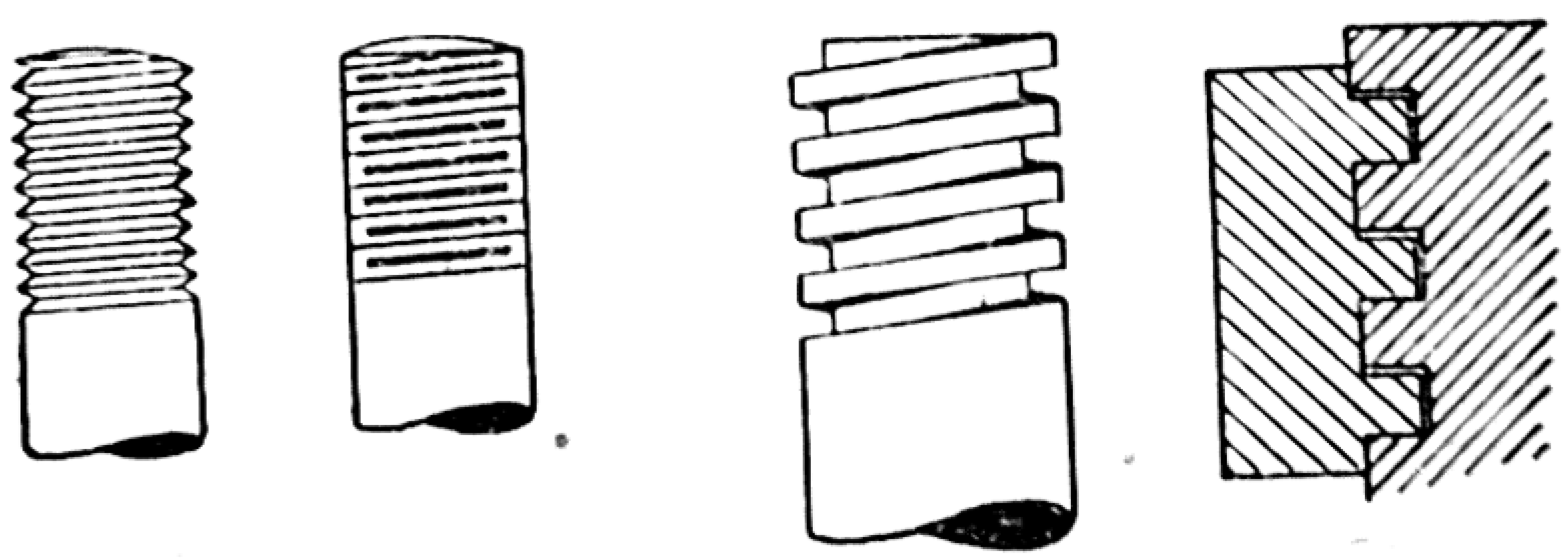


FIG. 1·5.—Micrometer Screw Gauge.

tube A, Fig. 1·5, this latter carrying a curved arm BC. Inside the tube A moves an accurate screw S, the pitch of which is 0·5 mm. —the pitch of a screw is equal to the length through which

the screw moves when it is rotated once about its axis. This movement is obtained by rotating the head H. This rotation also causes the collar K to turn around its own axis. The bevelled end of K is divided into 50 equal divisions, so that a rotation of K through one division corresponds to a movement of $\left(\frac{1}{50}\times 0{\cdot}5\right)$ mm. $= 0{\cdot}01$ mm., these divisions being used for interpolating the distance between the mm. divisions on A. The extremity of S and the face of C are perpendicular to the axis of the screw; between these two jaws the object to be measured is placed. When these jaws are in contact the zero on the bevelled edge should coincide with the zero on the mm. scale; before using the instrument this point should always be tested, and if the instrument has a zero error the corresponding correction [1] must be applied. The head H is arranged so that when the jaws of C and S are in contact, either with each other or some object, a further rotation of H fails to impart any movement to K.

Screws.—The micrometer screw gauge just described is an example of the use which is often made of an accurately cut screw. The threads of screws are generally triangular or square in section as in Fig. 1·6 (*a*) and (*c*). Perhaps the most important thread used



(*a*) (*b*) (*c*) (*d*)

FIG. 1·6.—Screws.

by engineers is the ***Whitworth*** **V** thread in which the angle of the thread is 55 degrees—cf. Fig. 1·7. In all screws the distance through which the screw advances when it makes one complete revolution is called the ***pitch*** of the screw. In diagrams screws are conventionally represented as in Fig. 1·6 (*b*).

Back-lash.—Screws may be used to impart a translatory motion

[1] The words ***correction*** and ***error*** are sometimes used as if they were synonymous. This is not so, it being preferable to define the correction as the quantity which must be ***added*** algebraically to the observed reading in order to obtain the true reading. The error is then equal to the negative value of the correction. Thus if a thermometer reads $-0{\cdot}6°$ C. when in melting ice, the temperature of which is defined as $0°$ C., then the correction is $+0{\cdot}6°$ C. and the error $-0{\cdot}6°$ C.

to a nut in which they work, and the amount of rotary motion which can be imparted to a screw without causing any movement of the nut is known as back-lash. It is sometimes due to wear or to imperfections in the manufacture. Very often, however, especially if the screw is intended for precision work, a certain amount of back-lash is allowed when the screw is being cut. The reason for this is that if an attempt is made to make the screw and nut fit exactly then the fit is soon destroyed by wear owing to the somewhat large forces operating upon screw threads.

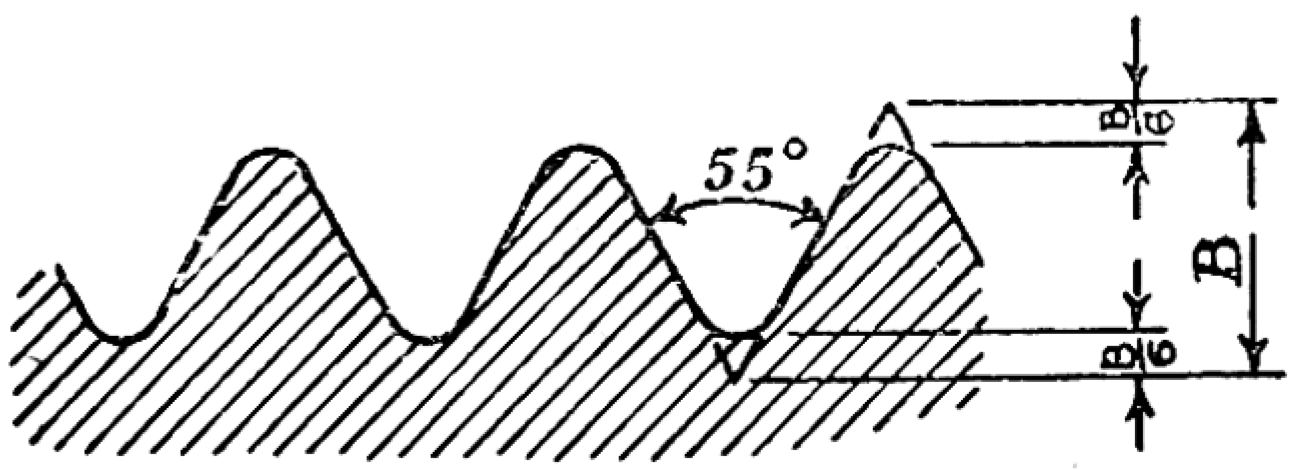


FIG. 1·7.—Whitworth Screw Thread.

It is therefore better to design the screw so that only one of its faces is in contact with the nut—in this way the wear is reduced to a minimum. When using screws for the purpose of estimating small distances, care must always be taken to turn the screw in one direction through a relatively large distance when setting the screw before an observation. Fig. 1.6 (*d*) will perhaps help to make these remarks more clear.

The Travelling or Vernier Microscope.—In order to measure short vertical or horizontal distances, a vernier microscope is frequently used. A precision form of this instrument is shown in Fig. 1·8. It consists of a microscope, M, clamped to a tube, A, supported in a rigid frame, B. When the microscope is thus clamped, a maximum displacement of 4 cm. may be imparted to it by means of a screw, S, operated by the milled head, H. The amount of this displacement is measured by a horizontal scale, D, which gives the complete number of revolutions of the milled head, the fractional part of a rotation being given by the divisions on the wheel, F, attached to the screw. The microscope may, however, be clamped in position at any point along the tube so that longer distances are measured by a succession of smaller displacements of the microscope. The microscope is fitted with an achromatic objective and eye-piece with cross-wires. The object under examination is supported on a small sliding table, resting upon geometric clamps, and provided with aligning adjustments operated by screws. A steel spring placed inside the tube, A, and attached to the stud, K (fixed to the stand, B), and to the end, L, of the tube, so that the spring is stretched, keeps the end, N, of the tube, A, in

contact with the extremity of the screw, S. [When the instrument is to be used to measure vertical distances, it is provided with a tripod base with levelling screws, so that the microscope may be traversed vertically.]

In order to measure with the aid of this instrument the diameter of a brass disc, for example, the eye-piece is first adjusted so that the cross-wires can be seen clearly—when this adjustment has been performed the eye-piece must not be disturbed again. The microscope is then raised or lowered by the rack and pinion, R, until the details in the object are visible. This having been done, the microscope is moved either horizontally or vertically until the image of the edge of the disc coincides with the cross-wire in the eye-piece, the microscope being adjusted so that there is no parallax. The appropriate scale is then read, and the microscope

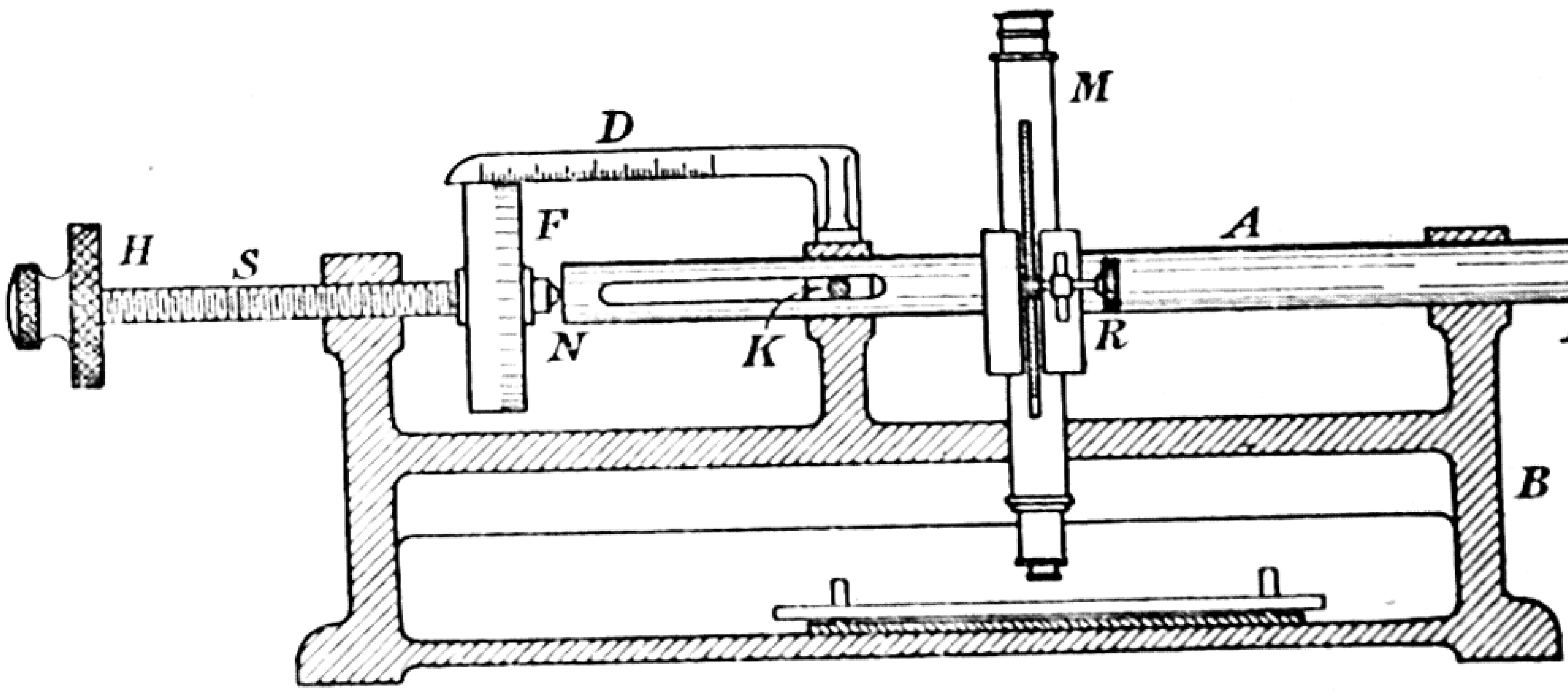


FIG. 1·8.—Travelling or Vernier Microscope.

afterwards moved so that the other extremity of the object coincides apparently with the cross-wire. The difference in the readings on the scale gives the diameter required.

Whenever it is necessary to determine the diameter of a circular object it is always advisable to measure two diameters in directions at right angles to each other. If the body has a cross-section which is slightly elliptical its mean diameter is the mean of any two diameters at right angles to one another, for it may be shown that the mean of any two mutually perpendicular diameters of an ellipse which is almost a circle is a constant for any given such ellipse.

Experiment.—The following exercise, which is to determine the number of centimetres equivalent to one inch, provides a means of becoming familiar with the use of a travelling microscope and illustrates also how to obtain a mean value from a series of observations. A steel scale graduated in inches and tenths of an inch is attached to the bed of the microscope and the microscope focussed on an image of one of the dividing lines on the inch scale, the eyepiece having previously been adjusted so that the cross-wires in the microscope are

clearly seen. The microscope is arranged so that there is no parallax between the cross-wires and the image of the particular dividing line which is being observed. The reading on the centimetre scale attached to the microscope is noted. The next dividing line is then observed and a similar reading obtained. The process is continued until about ten observations have been obtained. The fractional part of a centimetre corresponding to one-tenth of an inch may be found as follows. Let us suppose that when the microscope is moved by successive increments (tenths of an inch), the corresponding readings on the scale of the travelling microscope are $a_1, a_2, a_3, \ldots a_{10}$ (say). How are we to obtain arithmetically the best value for the shift of the microscope corresponding to 0·1 inch? If we deduce $(a_2 - a_1)$, $(a_3 - a_2)$, etc., and then calculate the mean of these quantities we only utilize the first and last observations, for

$$(a_2 - a_1) + (a_3 - a_2) + \ldots + (a_{10} - a_9) = (a_{10} - a_1).$$

This may be avoided by calculating

$$(a_6 - a_1), (a_7 - a_2), \ldots (a_{10} - a_5).$$

The mean of these quantities, viz.,

$$\tfrac{1}{5}[(a_6 + a_7 + \ldots a_{10}) - (a_1 + a_2 + \ldots a_5)]$$

then gives the average length in centimetres equivalent to 0·5 inch, and we notice that the calculation involves each reading once, and once only.

The Spherometer.—This instrument, which was specially designed for determining the radii of curvature of spherical surfaces, is shown in Fig. 1·9 (*a*). The circumference of the screw head H is divided into 100 equal divisions; the pitch of the screw is 0·5 mm., so that each division corresponds to 0·005 mm. The scale S gives the number of complete revolutions which the head makes. The points of the legs A, B, C are all in one plane and form an equilateral triangle. To set the instrument when, for example, the radius of curvature of a convex surface is being determined, it is placed on a sheet of glass and the screw-head, H, turned until D, the central leg of the instrument, is a little lower than A, B and C, Fig. 1·9 (*b*); if the instrument is tapped gently it rocks about an axis passing through the points of contact with the surface of D and of one of the fixed legs. The screw-head N is moved until this rocking ceases—the points A, B, C and D are then all in one plane. In estimating the position where the rocking just ceases, rotate the head through five divisions at a time when the position is being approached. Do this until the rocking ceases. Then, having rotated the head back through several divisions to avoid errors due to back-lash when the screw is advanced, turn the head in the initial direction through one division. Test for rocking on each occasion and proceed until the exact position of no rocking is located between two successive readings of the position of the head H. H is then raised until, when the spherometer is placed on the spherical surface, the instrument just ceases to rock. The

difference in the readings gives the height through which the screw D has been moved—let this be h. Let a be the distance between the outer legs and D when all four legs have their extremities in one plane. Then in Fig. 1·8 (c)

$$\mathrm{OA}^2 = \mathrm{AD}^2 + \mathrm{OD}^2$$

i.e. $$\mathrm{R}^2 = a^2 + (\mathrm{R} - h)^2 = \mathrm{R}^2 - 2\mathrm{R}h + h^2 + a^2$$

i.e. $$\mathrm{R} = \frac{a^2}{2h} + \frac{h}{2}.$$

If s is the distance AC, $s = a\sqrt{3}$, so that

$$\mathrm{R} = \frac{s^2}{6h} + \frac{h}{2}.$$

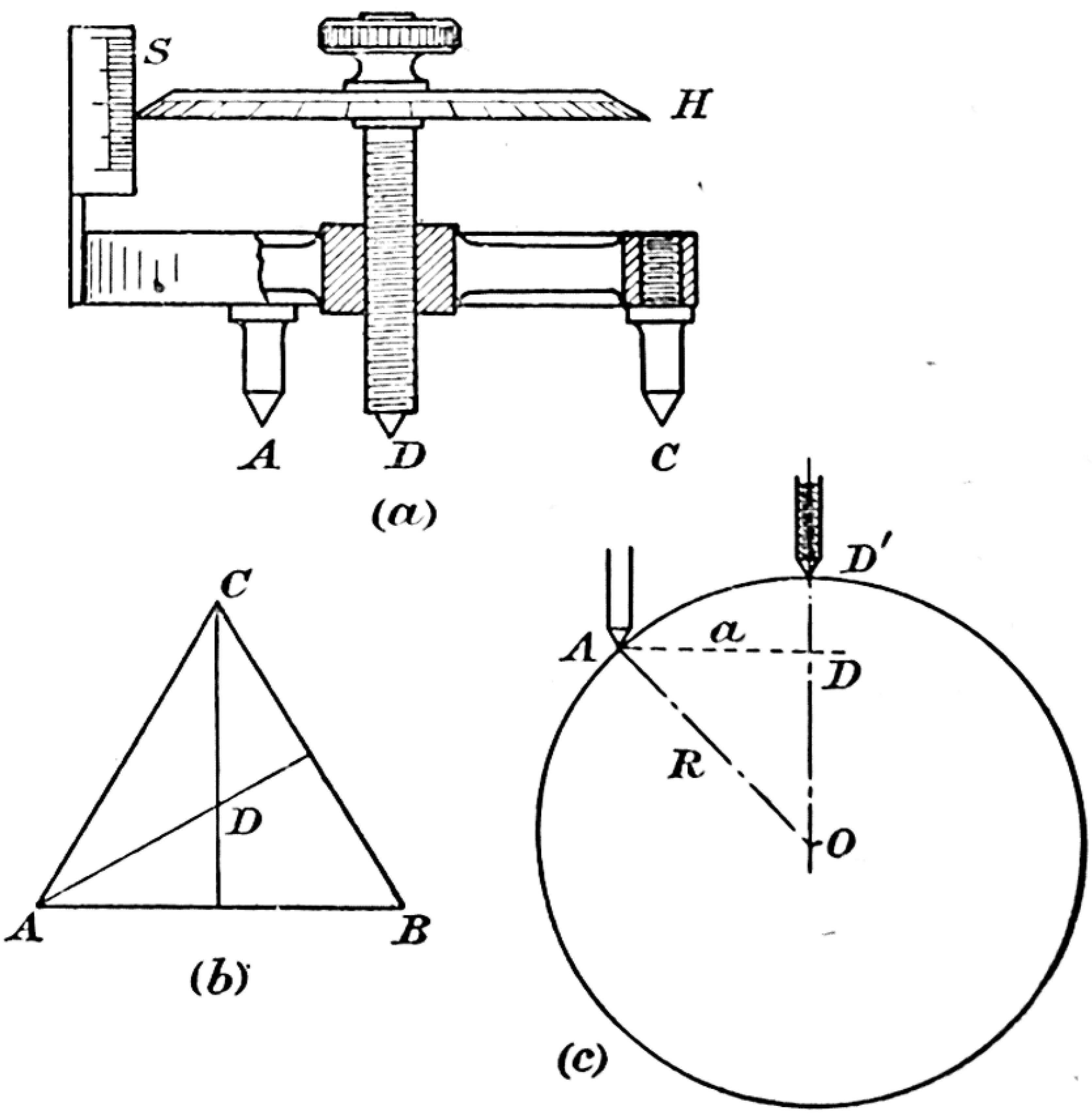


Fig. 1·9.—The Spherometer.

In using this instrument it is convenient to obtain first a reading with the central leg in the lowest position which it will occupy in any experiment, for, unless this procedure is adopted, the fractional parts are (1 − the reading on H); this tends to be confusing. The head is then rotated so that the screw D moves upward. The number of complete revolutions made by this head is best determined by counting, since it is not always easy to locate the position of the periphery of H on the vertical scale S. The fractional part of a rotation is determined in the usual way.

Example.—In determining the thickness of a piece of glass with a spherometer the pitch of whose screw is 0·5 mm., the readings were (i) 0·47 mm., (ii) three complete turns and a fractional part 0·25 mm.

$\therefore$ Thickness $= [(3 \times 0{\cdot}5) + 0{\cdot}25] - 0{\cdot}47$ mm. $= 1{\cdot}28$ mm.

Circular Verniers.—When it is essential to measure angles with an accuracy greater than that obtainable with a protractor, use is made of a circular vernier. The actual vernier found on any particular instrument will depend upon the smallness of the divisions on the main scale. As an example let us assume that this scale reads directly to half a degree. If the ultimate aim is to measure an angle correct to one minute the following procedure may be adopted. Twenty-nine divisions on the main scale are divided into thirty equal small divisions, so that the difference between one scale and one vernier division is one-thirtieth of half a degree, i.e. one minute. Hence if the 12th division on the vernier coincides with a division on the main scale, the fraction of a degree to be added to the reading of the main scale is 12 minutes. [Note that the main scale is divided into half-degree divisions.]

Indirect Methods of Measuring very great Distances.—Hitherto only direct methods of measuring a length, i.e. methods involving the repeated application of a standard or sub-standard rod of known length, have been mentioned. Let us see whether it is legitimate to apply indirect methods to measure lengths, such as the distance between two mountain peaks or that between the earth and a heavenly body. In these indirect methods a base line is selected—it may be the diameter of the earth's orbit—and various angles are measured. The required distance is then calculated by means of some trigonometrical formula. In this method certain light rays are identified with straight lines defining the sides of a triangle: but this is an assumption, finally to be tested experimentally. It is now known that for terrestrial distances the assumption is justified, but in an astronomical survey involving immense distances certain corrections have to be applied. These remarks are made here in order to show that there may be inherent difficulties when indirect methods are used to determine an apparently simple physical quantity such as a length.

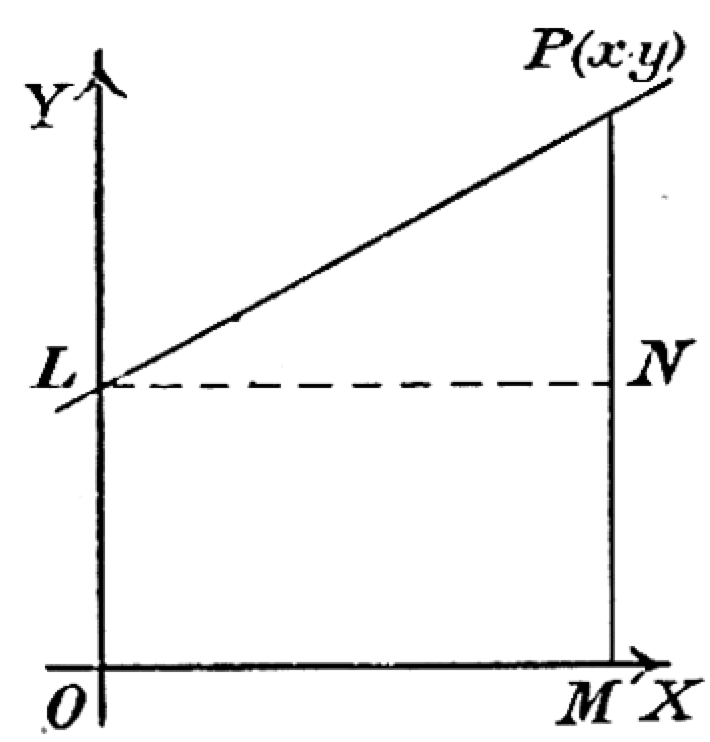


FIG. 1·10.

The equation $y = a + bx$.—Let PL, Fig. 1·10, be a straight line making an intercept of length a on the axis OY. Let P be any

point $(x, y,)$ on this line. Draw PM perpendicular to OX and LN parallel to OX. Then

$$y = \mathrm{PM} = \mathrm{PN} + \mathrm{NM}$$
$$= \mathrm{PN} + \mathrm{OL}$$
$$= \frac{\mathrm{PN}}{\mathrm{LN}} \cdot \mathrm{LN} + \mathrm{OL}$$

If we call $\frac{\mathrm{PN}}{\mathrm{LN}} = b$, this equation becomes

$$y = a + bx.$$

Any equation of this type therefore represents a straight line, i.e. it is a ***linear equation*** between the variables x and y; if an equation contains powers of x it may be said at once that it is the equation to some form of curve. The constant b in the above equation measures the ***slope*** of the line.

The Graphical Determination of Laws.—Whenever possible the results of an experiment should be shown graphically and if the results happen to lie on a straight line its equation determined. The constants in such an equation will often convey useful information to us. Moreover, if all except one or two of the points lie on a straight line, then such a graph tells us what observations should be repeated, for they are most likely to be in error. If the points do not lie on such a line, a fact which is best revealed by stretching a piece of black cotton across the paper, it may be advantageous to plot the logarithms of one or both of the quantities involved. The method of attack in such an instance may be gathered from the following example.

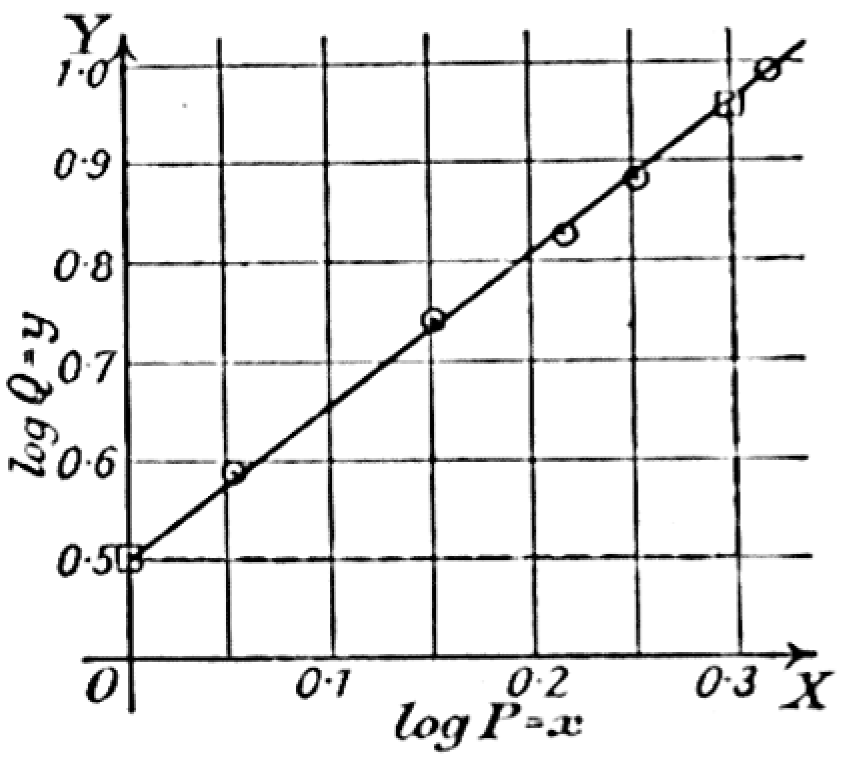


FIG. 1·11.

Example. The following numbers were obtained in a certain laboratory experiment.

Q	3·84	5·43	6·80	7·56	9·82
P	1·13	1·42	1·66	1·80	2·11

Discover the law connecting Q and P.

The graph obtained by plotting Q, as ordinate, against P is not a straight line. If, however, log Q is plotted against log P, as in Fig. 1·11, it is apparent that these two quantities are related to each other by a linear law. The intercept on OY is 0·50, while the slope is 1·51. [In measuring the slope of a line always choose two points on the line as far apart as possible—shown ⊡.] If we call $\log P = x$, and $\log Q = y$, the equation to the line is $y = 0{\cdot}50 + 1{\cdot}51x$, i.e. $\log Q = 0{\cdot}50 + 1{\cdot}51 \log P$.

The relationship between Q and P is therefore

$$Q = 10^{0 \cdot 50}P^{1 \cdot 51}$$

or
$$Q = 3{\cdot}18P^{1 \cdot 51}.$$

The Measurement of Mass.—The mass of a body is usually found by comparing that body with a set of standard masses, invariably referred to as a 'box of weights.' The comparison is carried out by means of a balance, which is really an equi-arm lever poised about a fulcrum. The masses to be compared are placed in pans which are suspended on knife-edges from the extremities of the beam. The accuracy of the balance depends, to a large extent, upon the design of the beam, which must be light but rigid. It must be light in mass if the sensitivity of the balance is to be high, and yet sufficiently rigid that its shape is not deformed under the greatest load for which the balance has been designed. The knife-edges are usually made of agate, this substance being chosen on account of the facts that it is hard, does not tarnish, and may be worked until a straight edge has been obtained.

The equality of the masses is ascertained by observing the deflections of a long vertical pointer perpendicular to the beam. When this pointer swings through the same distance on either side of its zero position, then the two masses in the pans are equal. The balance is protected in a glass case and the humidity of the atmosphere inside the case is greatly minimized by the use of concentrated sulphuric acid or solid calcium chloride contained in a glass receptacle. When the balance is not in use the beam and pans are not free to move, the beam being raised so that there is no permanent load on the knife-edges, whilst the pans rest on supports, all these conditions being obtained by the rotation of a small handle or wheel, which is outside the balance case.

Measurement of Time.—The evolution of watches and clocks is a result of man's desire to divide the mean solar day into smaller intervals of time. Most clocks depend upon the motion of a pendulum which, in its most simple form, consists of a heavy bob fixed to the end of a string, the other end being attached to some definite point. If the size of the bob is small compared with the length, l (cm.), of the string, then the time, T (sec.), of a complete swing, i.e. the time which elapses between the successive transits of the bob in the same direction past some fixed point or line, is given by the equation [cf. p. 38]

$$T = 2\pi\sqrt{\frac{l}{g}},$$

where g is the intensity of gravity [981 cm. sec.$^{-2}$ in England]. This equation enables the length of a seconds pendulum to be found,

this being the special pendulum making half a complete swing each second so that $T = 2$ sec. Its length is given by

$$l = \frac{T^2 g}{4\pi^2} = 99{\cdot}3 \text{ cm.}$$

Errors of Observation.—In this introductory chapter some remarks should be made concerning the accuracy of one's experimental results. The instruments available for determining the quantity in question should be critically examined to see what accuracy they can give, and care must be exercised to see that the final result recorded does not express an accuracy beyond the limits which the instruments can give. Suppose, for example, that the dimensions of a rectangular piece of wood are found by one student to be 3·96 cm. × 4·72 cm. × 1·74 cm. He may state that the volume is 32·522688 cm³. Is this result justifiable? Let us suppose that a second student measures the same block with the same pair of callipers. His observations are 3·98 cm., 4·75 cm., and 1·71 cm. respectively. If he proceeds to calculate the volume as the first student did, i.e. without thinking what he is doing, he will obtain 32·327550 cm.³ Why the difference? It is simply because their observations, just like all other observations, are subject to error so that they were not justified in stating anything more than 32·5 cm.³ and 32·3 cm.³ respectively.

It must be emphasized that when the first student asserts that the volume is 32·5 cm.³, he implies that his result is accurate to within 0·1 cm.³, i.e. the error is ± 0·1 cm.³ If, for example, there were reason to suppose that the error were ± 0·4 cm.³, he should indicate it by writing the result as (32·5 ± 0·4) cm.³

These few remarks may be further exemplified by extracts from the writings of a correspondent to *The Times*, 22 February, 1928. A few days before the result of a speed record had been quoted as 206·95602 m.p.h., a figure which implies that the speed was more than 206·95601 and less than 206·95603 m.p.h. If it does not, then the last figure has no meaning. To use this figure means that the error in the observations did not exceed one part in 20,695,602; thus, if the speed were measured over a distance of one mile exactly, then the timing gear gave results accurate to within one-millionth of a second. If, on the other hand, the timing gear was absolutely accurate, the distance of one mile must have been accurate to one three-hundredth of an inch. The correspondent ends by saying that a round figure of 207 m.p.h. is about the utmost that the actual measurements are likely to justify.

Errors due to Parallax.—The accuracy of the observations, for example, of the positions of the two points, whose distance apart is required, made with an ordinary scale graduated in cm. and

mm. is limited by the fact that the graduation marks have a finite thickness and because the eye cannot estimate accurately differences of length less than 0·1 mm. The error introduced in the measurement of a length in this way is likely to be at least 0·2 mm. Frequently, however, the observations may be much less accurate unless precautions have been taken to eliminate 'errors due to parallax'—i.e. to the apparent change in the position of an object with reference to some fixed object due to a change in the position of the observer. Such errors arise in the present instance if the scale is used as in Fig. 1·12 (*a*) owing to the thickness of the bar on which the scale is engraved. Thus, with the eye at E_1, the position of A appears to be 9·1 cm.; from E_2 it is 9·2 cm., etc. To eliminate such errors the graduated edge of the scale should be placed in contact with the points between which the distance is to be measured—cf. Fig. 1·12 (*b*).

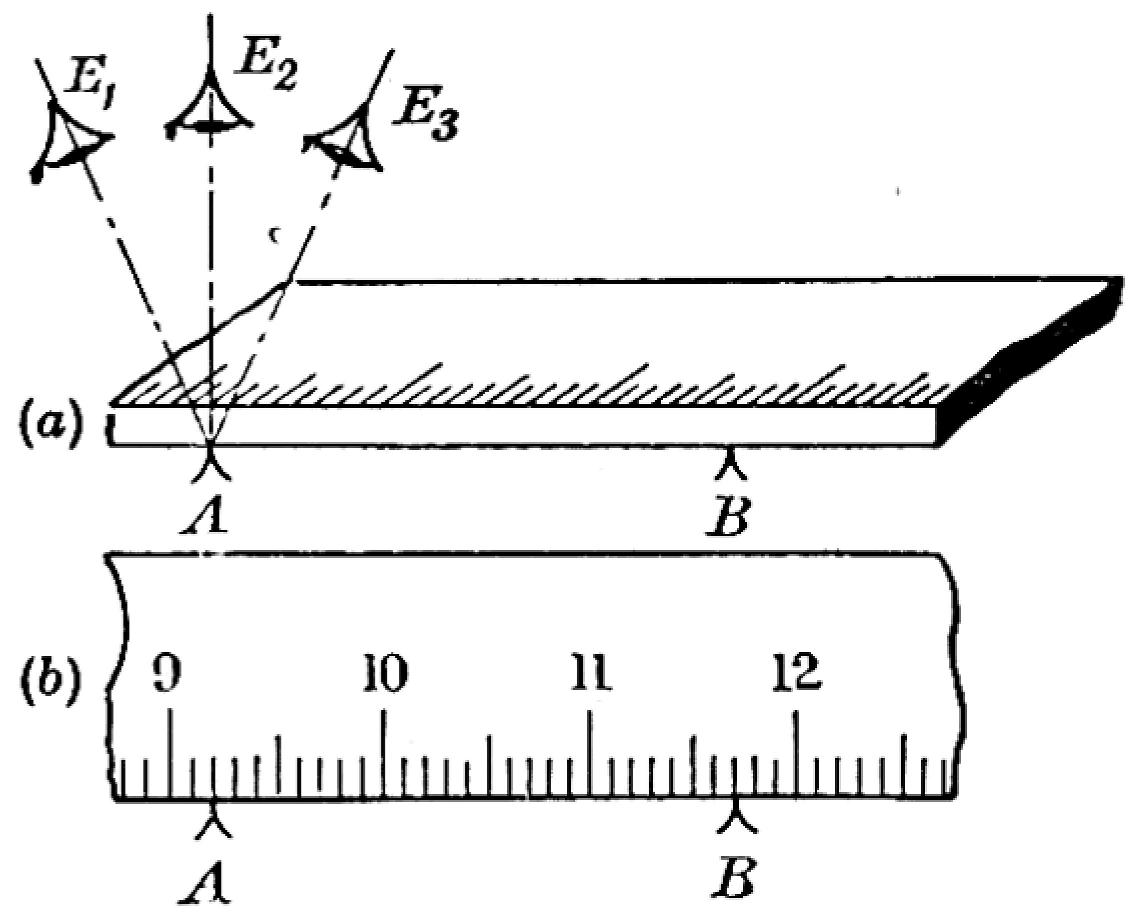


FIG. 1·12.—Errors due to Parallax.

EXAMPLES I

1.—Calculate the circular measure of an angle of 47°. By means of a diagram calculate its sine, cosine, and tangent. What is the angle whose circular measure is unity?

2.—Describe a vernier and its use on a pair of callipers.

3.—Describe, with the aid of a diagram, a micrometer screw gauge.

4.—If the diameter of the earth were increased by 1 ft., calculate the increase in its circumference.

5.—Calculate the length of a simple pendulum which will beat half-seconds.

6.—Criticize the following: A rectangular block measures 7·16 cm. × 6·73 cm. × 4·05 cm. Its volume is therefore 195·1565 cm.^{3}

7.—Discuss the advantages of representing a series of observations by means of a graph. How would you plot a series of observations of the time taken by a trolley to travel different distances down an inclined plane, so as to bring out the law involved as clearly as possible?

8.—Describe a spherometer and deduce the formula necessary when using this instrument to determine the radius of curvature of a spherical surface.

CHAPTER II

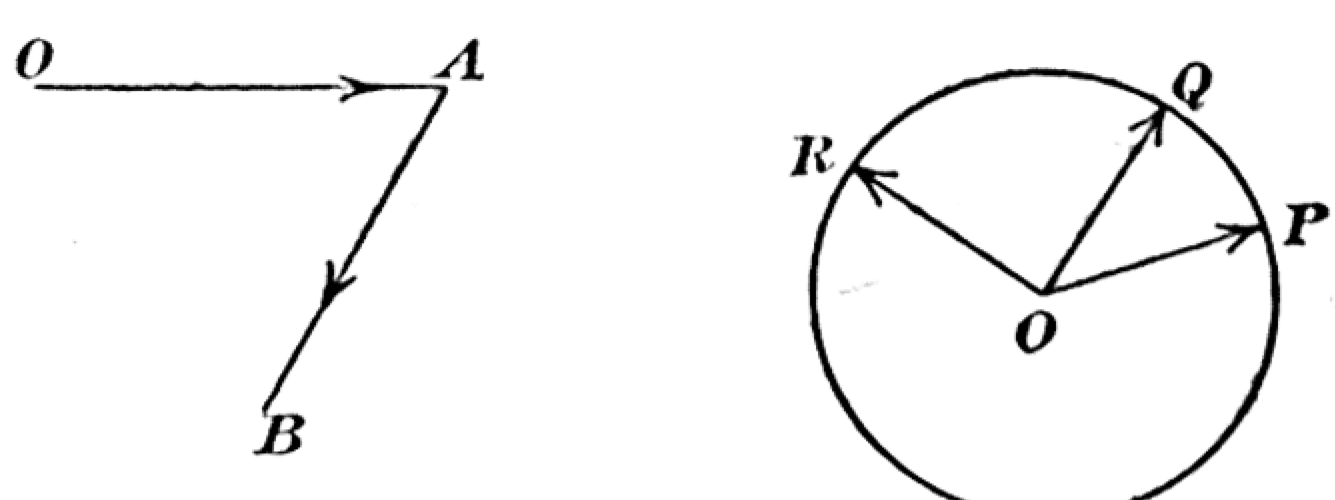
THE ELEMENTS OF DYNAMICS

Mechanics.—The science of mechanics deals with the properties of bodies, and it is usually studied under the headings, or sections, called ***Dynamics, Statics*** and ***Hydromechanics.*** The first branch, viz. ***Dynamics,*** deals with bodies which are in motion relative to their surroundings; the second, viz. ***Statics,*** concerns bodies at rest, whilst ***Hydromechanics*** is the science of liquids. This latter subject is again subdivided into ***Hydrostatics*** and ***Hydrodynamics,*** the former section dealing with liquids at rest whilst in ***Hydrodynamics*** the motion of liquids is studied.

The Material Particle.—At this stage, perhaps, reference ought to be made to the size of the objects we discuss in an elementary treatment of the subject of dynamics. The equations which are deduced only apply to a body which is so small that it may be regarded as a mathematical point—such a body is known as a ***material particle.*** If the body is not small, attention must be paid to the fact that it is capable of rotation and therefore possesses energy due to rotation. A material particle may therefore be defined as a body which is so small that its energy of rotation may always be neglected.

MOTION IN A STRAIGHT LINE

Displacement.—The motion of a body is only detected by observing its position with reference to its surroundings. If the position of the body changes with reference to its surroundings, then that body is said to be undergoing a ***displacement.*** Thus, if at some initial time, a body—here represented by a point—is in the position O, Fig. 2·1(*a*), and later on it is at A, then the body has been displaced, and the



(*a*) (*b*)

FIG. 2·1.

displacement is expressed by the length and direction of OA. At some later period in its history the body may be at B, so that its further displacement is AB, whilst the actual displacement from the origin is OB. The idea of displacement must always convey that of direction as well as that of magnitude. Thus, if O, Fig. 2·1 (*b*), is the original position of a body, whilst P, Q, R, etc., points on the circumference of a circle whose centre is O, are its subsequent positions, then the magnitude of the displacement is fixed but the direction is variable.

Sense of Direction.—Every displacement has magnitude and direction; they all have 'sense' too; for example, a body may be displaced from O to A, or from A to O. The sense of the direction is opposite in these two instances, the sense of the displacement being indicated in a diagram by the use of small arrow-heads.

Representation of Displacement.—A displacement is represented on a drawing by a straight line whose direction and sense are that of the displacement and whose length is proportional to the magnitude of the displacement. Any quantity which can be represented in magnitude, direction, and sense, is called a ***vector***. Other quantities are ***scalars***. Velocity, force, magnetic intensity, etc., are vectors, while potential, energy, money, etc., are scalars.

Relative Displacement and Rest.—If two trains are moving in the same sense along parallel tracks, a passenger in one of the trains may observe the following facts:—If he is travelling in the faster train, the other train will appear to him as if it were receding, whilst if he is in the train which is moving less rapidly, he will observe a forward motion of the faster train. ***Relative displacement*** is defined as the displacement of one body with respect to another. In fact all displacements must of necessity be relative ones, although we are accustomed to think of absolute displacement because our idea of rest is generally associated with non-moving bodies on the surface of the earth. Actually the earth is moving on its own axis, round the sun and through space, so that when it is said that a body is at rest, the statement is only intended to convey the fact that its displacement, with respect to its surroundings [generally on the surface of the earth], is zero.

The Composition of Displacements.—In order to fix our ideas let us consider the displacement of a marble which rolls across the floor of a moving carriage. Let PQRS, Fig. 2·2, be the carriage, while O is the initial position of the marble. Let us further assume that in the time required for the marble to travel from O to B the point in the carriage corresponding to O has moved to A, i.e.

A occupies the same position relative to the final position P′Q′R′S′ of the carriage as O did with respect to the initial position PQRS. If C is the point corresponding to B, it follows that C is the final position of the particle and that the actual displacement is completely represented by OC. We say that it represents the actual displacement, but it must be understood that this implies the existence of a reference body absolutely at rest. Such a body is unknown. The result just obtained is a particular instance of a general theorem—we refer to the ***parallelogram law of vectors*** which may be stated as follows:—***Two vectors of the same type may be added together by constructing a parallelogram***

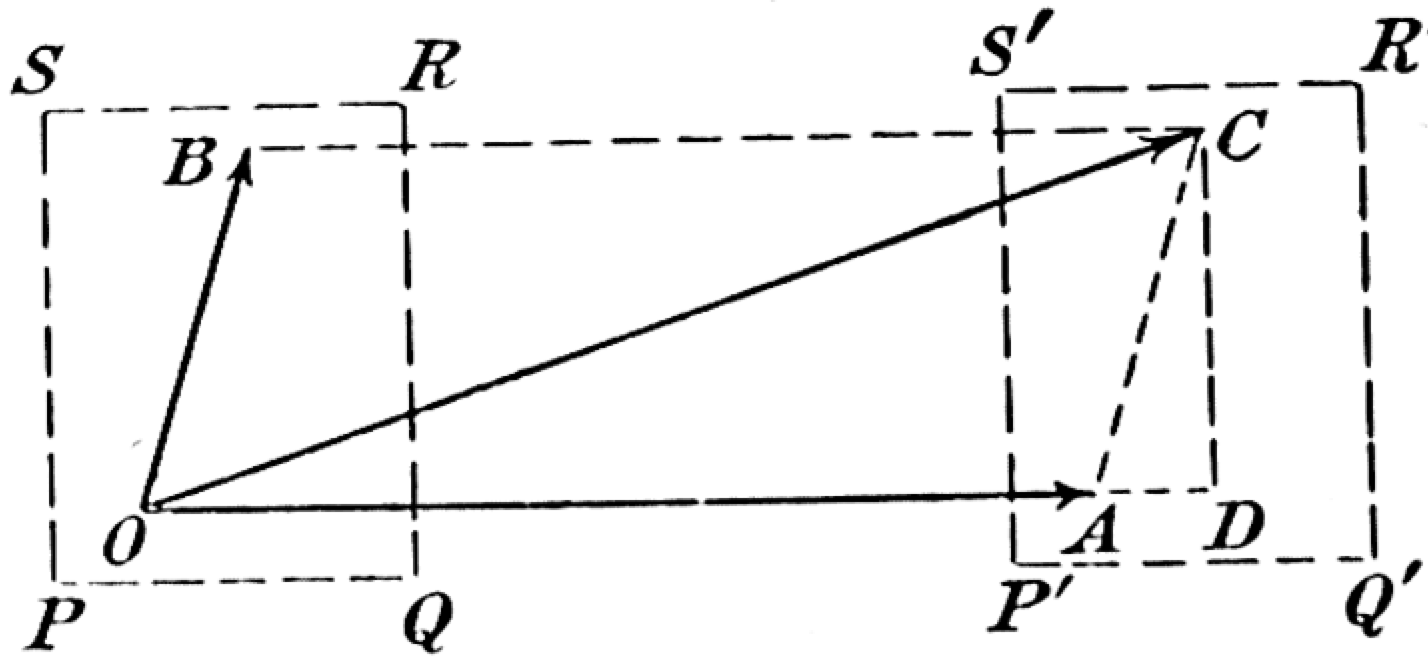


FIG. 2·2.—The Composition of Displacements (Vectors).

the adjacent sides of which are proportional to the vectors. The diagonal drawn through the point of intersection of these two sides represents the resultant of these two vectors completely. Thus OC, Fig. 2·2, is the resultant of the two like vectors OA and OB. The vectors OA and OB are said to have been added vectorially.

The magnitude of the resultant is easily found, for if CD is drawn perpendicular to OA to meet OA produced in D, then

$$\begin{aligned} OC^2 &= OD^2 + CD^2 \qquad [\because \widehat{ODC} = 90°] \\ &= (OA + AD)^2 + CD^2 \\ &= OA^2 + (AD^2 + CD^2) + 2 \,.\, OA \,.\, AD \\ &= OA^2 + AC^2 + 2 \,.\, OA \,.\, AC \cos\theta \\ &= OA^2 + OB^2 + 2 \,.\, OA \,.\, OB \,.\cos\theta, \end{aligned}$$

where θ is the $\widehat{AOB}$.

Speed.—The idea of speed is obtained by associating the conception of time with that of displacement. If, for example, a ship goes from one port to another in one day, while another occupies two days for the same journey, then the speed of the former vessel is twice that of the second. The speeds which are referred to here are average values of the speeds of the vessels, because the vessel starts from rest and comes to rest at some other point, so that its actual speed at some times will have been less than its average

speed, whilst at others it will have been greater. The average speed of a body is given by the expression

$$\text{average speed} = \frac{\text{distance traversed}}{\text{time occupied in so doing}}.$$

[Care must be taken to avoid such expressions as 'the speed of the ship was 22 knots per hour,' since ***knot*** is a nautical term used to imply a speed of one sea-mile per hour. A sea-mile is intended to be such a distance on the earth's surface that an angle of one minute is subtended by that arc at the earth's centre. The British Admiralty takes this to be 6020 feet.]

Velocity.—When a body moves in a definite direction the speed of the body in that direction is called its ***velocity***. It is important to remember that velocity always implies speed in a fixed direction.

Uniform Velocity. The term ***uniform velocity*** is used to convey the idea that the distance traversed in any small interval of time is the same for all such intervals, however small the interval of time may be. Thus, if a body moves in a given direction 20 ft. in 5 secs. it is not justifiable to say that its velocity is uniform, for it is conceivable that if the position of the moving object had been observed at the end of every second, say, then the displacements in those seconds may have been found, for example, to be 3, 5, 6, 4 and 2 ft. respectively. The velocity (strictly, the mean velocity) in the first second is 3 ft. sec.$^{-1}$; in the second second 5 ft. sec.$^{-1}$; in the third second it is 6 ft. sec.$^{-1}$, etc., whilst the average velocity over the interval is 4 ft. sec.$^{-1}$ But if the velocity had been uniform and the body had been displaced 20 ft. in 5 secs., then in 1 sec. the displacement would have been 4 ft.; in 0·5 sec. 2 ft.; in 0·25 sec. 1 ft., etc.

Velocity-time Curve or Graph.—Suppose that a particle has an initial velocity of 2 ft. sec.$^{-1}$ and that at the ends of the first 3 sec. of its motion its velocity is 3·4, 4·0 and 4·2 ft. sec.$^{-1}$ respectively. Its motion can be shown graphically as follows: axes OX and OY are chosen at right angles to each other; one division along OX representing 1 sec., while one division along OY represents a velocity of 1 ft. sec.$^{-1}$—cf. Fig. 2·3 (*a*). The point A indicates the initial velocity, while the points B, C and D indicate the subsequent velocities of the body. Now the points A, B, C and D can be joined together either by short straight lines [shown dotted] or, since it is legitimate to presume that the velocity of the body does not change abruptly but continuously, they may be joined together by means of a smooth curve. This smooth curve is called the ***velocity-time curve***.

It is now necessary for the significance of the area under this

curve, i.e. the area OABCDRQP, to be found and its meaning interpreted. Since velocity means displacement per unit time, the statement that the uniform velocity of a body is 10 ft. sec.$^{-1}$ implies that the distances traversed in 1, 2, 3 and 4 secs. of its motion will be 10, 20, 30 and 40 ft. respectively. In Fig. 2·3 (*b*) the velocity-time curve for such a motion has been constructed; this curve will be the straight line AE, because the velocity is constant. Now an area of 1 sq. unit represents a distance traversed of 5 ft., because the unit length parallel to OX represents 1 sec. and

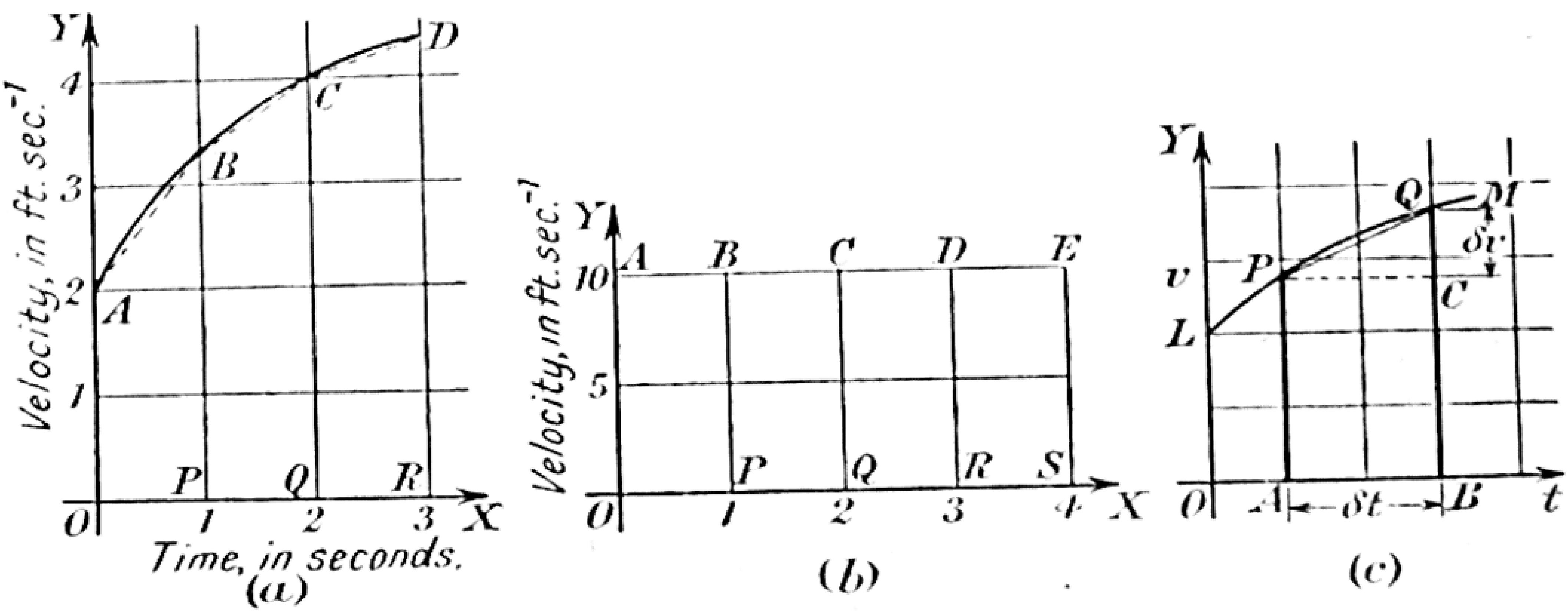


FIG. 2·3.—Velocity-time Curves.

the unit length parallel to OY represents a velocity of 5 ft. sec.$^{-1}$, so that the area OABP indicates a distance traversed of 10 ft., since OABP is 2 sq. units in area. The complete rectangle has an area of 8 sq. units, so that the distance traversed in 4 sec. is $8 \times 5 = 40$ ft., a value which agrees with that obtained from the definition of velocity [cf. p. 21].

Now when the area under the curve is an irregular one, as in Fig. 2·3 (*a*), it is still true that this area represents the distance through which the body has moved. In this particular example the area is 10·51 sq. units, and since 1 sq. unit represents a distance traversed equal to 1 ft., the actual distance traversed is 10·51 ft.

Algebraic Formula for Distance Traversed.—If the velocity of a body is uniform and equal to u ft. sec.$^{-1}$, the velocity-time curve will be a straight line parallel to the time-axis. If unit distance along the axis OX represents 1 sec. and unit distance along OY represents unit velocity, after t sec. the area of the velocity-time curve will be ut units of area, so that s, the distance traversed is ut, because unit area represents unit distance,

i.e. $$s = ut.$$

Acceleration and Retardation.—Consider the velocity-time curve obtained from the following observations :—

Time in seconds . . .	0	1	2	3	4	5
Velocity in ft. sec.$^{-1}$ . .	4·0	4·5	5·0	5·5	6·0	6·5

The graph is shown in Fig. 2·4, and it is at once apparent that the 'curve' is a straight line, i.e. it is linear. Whenever the velocity-time graph is linear, the motion is said to have been ***uniformly accelerated*** if the velocity has been increasing, and ***uniformly retarded*** if the velocity has been diminishing. These terms indicate that the velocity has been increased or diminished by equal amounts in equal intervals of time.

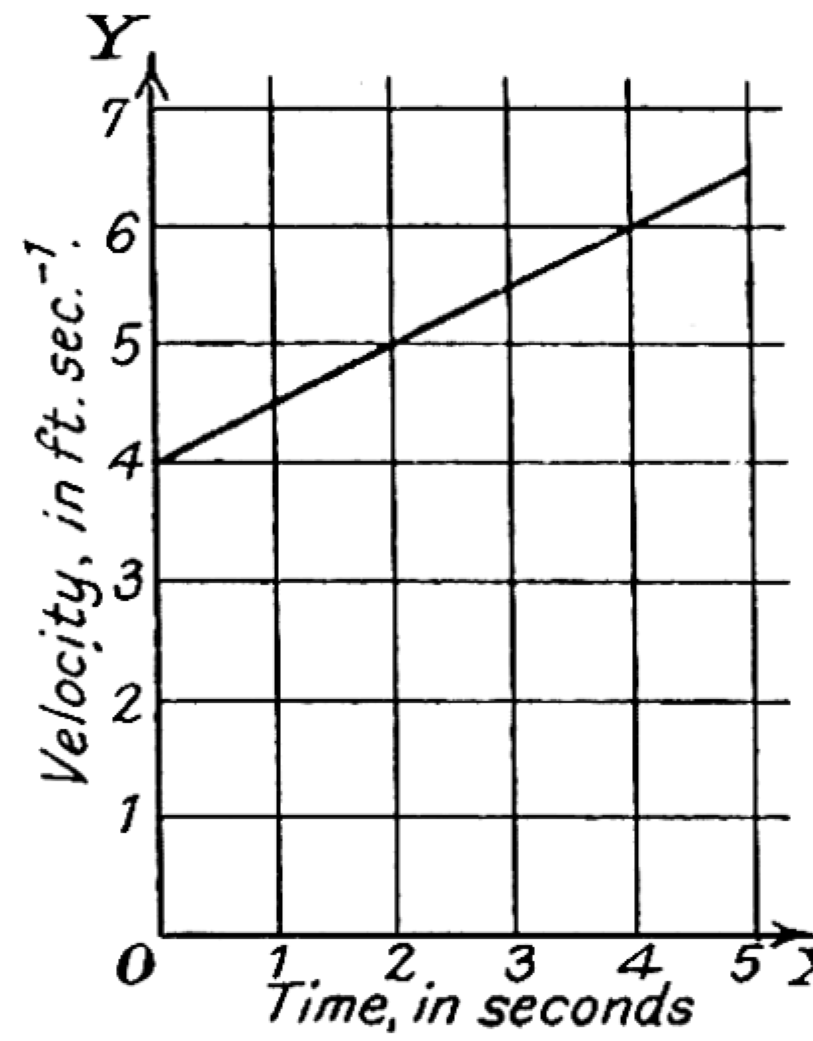


FIG. 2·4.—Velocity-time Curve for Uniformly Accelerated Motion.

Definition.—***Acceleration, i.e. the rate of change of velocity, is said to be uniform when the velocity increases by equal amounts in equal intervals of time, however small those intervals may be.***

A glance at the above table shows that the change in velocity is 0·5 ft. per sec. every second; or, as it is more usually written 0·5 ft. per sec. per sec., or 0·5 ft. sec.$^{-2}$.

When the velocity-time curve is not a straight line the velocity does not increase by equal amounts in equal intervals of time; i.e. the acceleration is ***non-uniform***. The acceleration at any particular instant may be determined as follows. Let P and Q, Fig. 2·3 (*c*), be two neighbouring points on a velocity-time curve LM. Draw PA and QB parallel to OY, and PC parallel to OX. Then PC and CQ represent small increments in the time and velocity respectively: we denote them by δt and δv. Since these two quantities are small their ratio $\frac{\delta v}{\delta t}$ is the mean acceleration of the moving particle during the interval of time when its motion is represented by P and Q. As the points P and Q move toward one another, the ratio $\frac{\delta v}{\delta t}$ remains finite and approaches a limiting value which is the acceleration of the particle at P. This limit is denoted by $\frac{dv}{dt}$, or $\dot{v}$, and measures the slope of the tangent to the curve at P. Thus, to determine the acceleration at a given time, the tangent

at the corresponding point on the velocity-time curve must be drawn. Its slope gives the acceleration required. In general, it is difficult to draw a tangent accurately, so that to obtain the acceleration at any instant we must measure the slope [acceleration] of the curve at several points and construct a curve showing the relation between the slope and the time. When this curve has been drawn the acceleration at any instant may be read off at once and the value obtained will be more reliable than that found by drawing the tangent at the point in question, since several tangents have been drawn in constructing the final curve.

Angular Velocity and Angular Acceleration.—When a particle, P, is rotating in a plane about a fixed point O the rate at which the angle between OP and a fixed line OX changes is termed the ***angular velocity*** of the point P. It is generally denoted by the symbol ω. The rate of change of the angular velocity is called the ***angular acceleration***, (α), i.e. $\alpha = \omega$.

Algebraic Formulæ for Uniformly Accelerated Motion

To find the Velocity after a Given Time.—Let u be the initial velocity, a the acceleration, t the time and v the final velocity of a moving body, all these quantities being expressed in the same system of units. Now a, the acceleration of the moving body, is the increase in velocity per unit time, so that the velocity at the end of the first unit time interval is $u + a$. Similarly the velocity at the end of the second unit interval will be $(u + a) + a$, or $u + 2a$. Hence, at the end of time, t, the velocity will have increased by an amount at, so that the actual velocity which has already been called v is then $u + at$.

$$\therefore v = u + at.$$

To find the Space Traversed in a Given Time Interval.—If the motion of the particle is represented by the symbols used in the last paragraph, the velocity-time graph is easily constructed; since the acceleration is uniform, we know from the definition of such an acceleration that the graph must be a straight line. This is represented by AB in Fig. 2·5. If BC is drawn perpendicular to the axis OX the area OABC represents the space traversed in time OC $= t$.

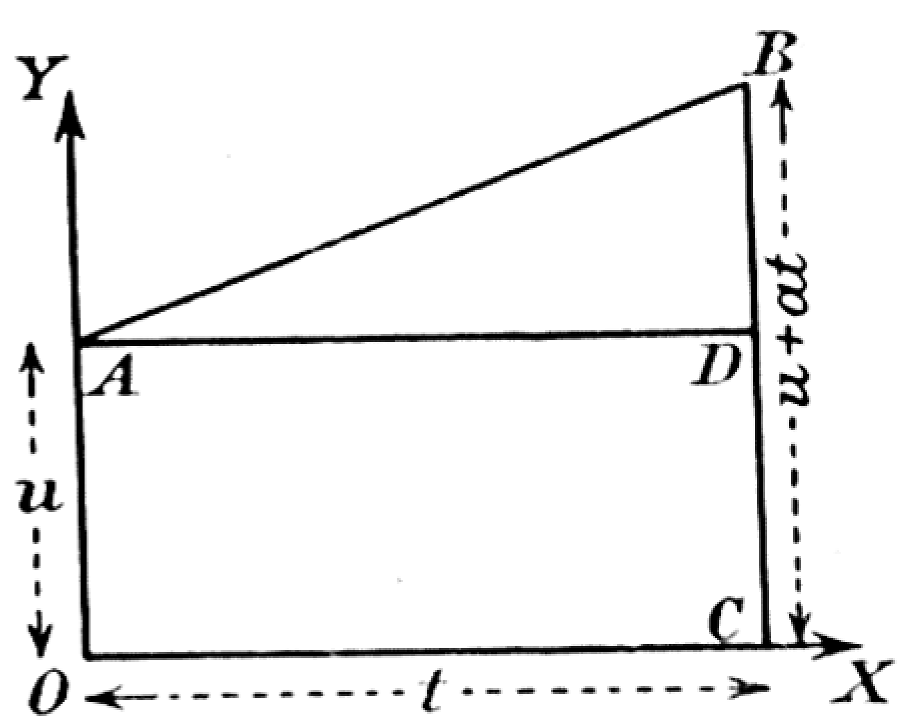


Fig. 2·5.—Velocity-time Curve for Uniformly Accelerated Motion.

$$\begin{aligned}\text{Now area OABC} &= \text{rect. OADC} + \Delta\,\text{ABD} \\ &= \text{OC} \,.\, \text{CD} + \tfrac{1}{2}\text{AD} \,.\, \text{DB} \\ &= ut + \tfrac{1}{2}at^2\end{aligned}$$

and this is s; consequently $s = ut + \frac{1}{2}at^2$.

The two equations which we have proved can be combined to form a new one. We have

$$\begin{aligned}v^2 &= (u + at)^2 \\ &= u^2 + 2a(ut + \tfrac{1}{2}at^2) \\ &= u^2 + 2as.\end{aligned}$$

Motion under Gravity.—When a body moves either towards or from the surface of the earth it is said to be moving under the influence of gravity. Such motions have been considered from the days of the Greeks, foremost among whom was ARISTOTLE, who taught that heavier bodies fell towards the earth more rapidly than lighter ones. The validity of this doctrine was not disputed until GALILEO [1564–1642] showed that two bodies, the mass of one being ten times that of the other, fell together when released at the same instant. These experiments, which were conducted from the leaning tower of Pisa before the eyes of his opponents, may be said to have sounded the 'last post' over the old doctrines which were founded on speculation, and which, in time, have been superseded by teachings based upon experimental fact. The ideas of Galileo were developed by SIR ISAAC NEWTON [1642–1727], who devised the so-called ***guinea-and-feather experiment***. He allowed a guinea and feather to fall in air and showed that the guinea reached the ground first. The cause of this apparent exception to the teachings of Galileo was discovered by performing the experiment in an exhausted tube. Under such conditions the feather and guinea fell together, for there was then no resisting medium to retard the motion of the feather.

The Acceleration of Falling Bodies.—All bodies fall towards the earth with a constant acceleration g,[1] which is equal to 32 ft. sec.$^{-2}$, or 981 cm. sec.$^{-2}$ in these latitudes. The value of g varies at different places because the earth is neither a true sphere, nor is it homogeneous. In addition, the earth's surface is not smooth, so that the variation of the intensity of gravity with altitude always has to be considered.

When a body moves under the influence of gravity its motion is determined by the equations,

$$\begin{aligned}v &= u \pm gt \\ s &= ut \pm \tfrac{1}{2}gt^2 \\ v^2 &= u^2 \pm 2gs.\end{aligned}$$

[1] 'g' is now termed the *intensity of gravity*—cf. p. 27.

The plus sign is used for falling bodies, the negative for those which rise, and the distance s is considered positive when it is measured vertically upwards. [For bodies starting from rest, $u = 0$.]

Momentum and Force.—Bodies only move relatively to their surroundings if they are acted upon by some ***external agency***, and by experience we know that it is more difficult to move some bodies than others. This is because the bodies have different masses, where ***mass*** is defined as the quantity of matter in a body. To this statement must be coupled the idea that two masses are equal if, when moving with equal velocities but in opposite senses, they are reduced to rest after a collision in which there is no rebound. The external agency which is capable of imparting motion to a body is called ***force***. Now when a force acts on an object it cannot increase the mass of the body, and yet we know that the larger the force which acts on a body, the more difficult it becomes for the motion to be arrested. It is the ***momentum*** of the body which has been increased by the larger force, and it continues to be increased by the force during such time that the latter is operative. The momentum I is defined as the product of the mass, m, of the body and its velocity v, so that $I = mv$.

Newton's Laws of Motion.—

LAW I.—***Every body continues in its state of rest or uniform motion in a straight line, unless impressed forces are acting upon it.***

LAW II.—***Change of momentum per unit time is proportional to the impressed force, and takes place in the direction of the straight line along which the force acts.***

LAW III.—***Action and reaction are always equal and opposite.***

GALILEO discovered the first two laws quoted above towards the end of the sixteenth century, whilst the third was known to HOOKE, HUYGHENS and WREN. The three laws were formally stated by Newton in his ***Principia*** in 1686.

A formal proof, analytical or experimental of these laws is not possible, but on them is based the whole system of dynamics, including astronomy. Since the results obtained and the predictions made by astronomers are in good accord with facts, it becomes difficult to imagine that the laws on which their arguments finally depend are erroneous.

Force.—Newton's second law provides us with a means of measuring forces. The proportionality implied in the law may be made into equality by an appropriate choice of units. If the

velocity of a body changes from v_1 to v_2 in time t, the force F is given by

$$F = \frac{m(v_2 - v_1)}{t} = ma \qquad \left[\because a = \frac{v_2 - v_1}{t}\right],$$

only when this particular choice of units has been made ; the force then is equal to the change of momentum per unit time.

Units of Force.—When the mass of a body is given in pounds, and the acceleration in feet per second per second, the force is expressed in ***poundals.*** The absolute unit of force in the f.p.s. system is the ***poundal,*** which is defined as ***that force which, acting on a body of mass 1 lb., will impart to it an acceleration of 1 ft. sec.***$^{-2}$. In the c.g.s. system the absolute unit of force is the dyne, which is ***that force which, acting on a mass of 1 gm., will impart to it an acceleration of 1 cm. sec.***$^{-2}$.

Mass and Weight.—Whenever a body of mass m moves under the influence of gravity it acquires an acceleration g. The magnitude of the force which causes this motion is mg, and it must be attributed to the attraction which the earth has for all matter. This force is called the ***weight*** of the body. We can therefore find the ***weight*** w, of a ***mass*** of 1 lb. It is $w = 1 \times g = 1 \times 32 = 32$ poundals. Hence a mass of 1 lb. has a weight of 32 poundals, by which statement it is to be understood that the force due to gravity on a 1-lb. mass is 32 poundals.

Engineers find that the poundal is too small a unit of force for practical purposes and so choose one which is 32 times as large. This is the pound-weight unit—abbreviated to 1 lb.-wt. The statement that a force of 6 lb. acts on a body has no meaning. The idea which it is intended to convey is that the force is equal to that which is operative on a 6-lb. mass due to gravity, so that its magnitude is (6×32) poundals or $\left(\frac{6 \times 32}{32}\right) = 6$ lb.-wt., the correct statement being that a force of 6 lb.-wt. acts on the body.

Absolute and Gravitational Units of Force.—A poundal and a dyne are termed ***absolute units of force,*** since their values are independent of g, the acceleration due to gravity, a quantity which varies at different places. A pound-weight (1 lb.-wt.) and a gramme-weight (1 gm.-wt.) are called ***gravitational units of force*** since they depend on the value of g.

The Intensity of Gravity.—The force due to gravity acting on a small mass δm near to the earth's surface is $\delta m.g = \delta F$ (say). The force per unit mass is therefore $\frac{\delta F}{\delta m} = g$. From analogy with

the definitions of the strengths of electric and magnetic fields, g is termed the ***intensity of gravity.***

Newton's Third Law of Motion.—Let us examine the statement 'Action and reaction are always equal and opposite' in more detail.

When a book rests on a horizontal table, the action or thrust of the book on the table is equal to the reaction or thrust of the table on the book. In this instance, the action and reaction must be equal and opposite, for if not, motion would ensue. This example, dealing with bodies at rest, presents no difficulty. But Newton's third law of motion postulates that action and reaction are always equal, i.e. even when the two bodies move. Newton considers the particular instance of a horse drawing a cart, 'If action and reaction are equal and opposite, how is it that the horse and cart move forward?' is a question not infrequently asked.

Before attempting to solve this difficulty, which is often a very real one, it must be emphasized that in attempting to solve any mechanical problem, the first essential thing is to fix upon the system whose rest or motion is to be discussed. The system may comprise one body, several bodies, or many bodies, but the system must be clearly defined before the solution is attempted.

In the present problem three possibilities for discussion are as follows:—(i) the motion of the cart, (ii) the motion of the horse, and (iii) the motion of the horse and cart together. If we begin with the first then we must imagine the cart to be isolated from the horse and all other objects by some imaginary closed surface. The forces acting on the cart are

(*a*) an attraction due to the earth—this is called the weight of the cart;

(*b*) an upward thrust (the resultant of the thrusts at the points where the wheels are in contact with the ground).

But by the third law of motion to each of the forces (*a*) and (*b*) equal and opposite forces are exerted by the cart on the earth. These are no concern at present, however, for we have agreed to discuss the motion of the cart, and accordingly have isolated it and have to consider the forces acting on the cart only.

Now the forces (*a*) and (*b*) must be equal and opposite, for if not, the cart would rise or fall according as the thrust of the earth on the cart were greater or less than the attraction of the earth on it. Since there is no motion of the cart in a vertical direction these forces balance, and we do not have to consider them further.

(*c*) Let T be the tension in the traces.

(*d*) Let R be the force due to friction, air resistance, etc. Then if $T > R$ the cart will move forward, its acceleration being

$$\frac{T - R}{M_1} = a_1 \text{ (say)}$$

where M_1 is the mass of the cart.

Similarly, if the motion of the horse is discussed, we shall find that the external forces (not balanced) acting on him are

(i) the tension in the traces (acting backwards),

(ii) the horizontal component, F, of the thrust of the earth on the horse.

If $F > T$ the horse moves forward with an acceleration a_2 given by

$$a_2 = \frac{F - T}{M_2}$$

where M_2 is the mass of the horse.

Since $a_1 = a_2$, the common acceleration of the horse and the cart is

$$\frac{F - R}{M_1 + M_2}$$

If we consider the horse and cart together, the unbalanced forces are F and R, the acceleration being

$$\frac{F - R}{M_1 + M_2}$$

as before. In this instance it is not necessary to consider the tension, since it is now only an internal reaction between two parts of the system; this cannot affect the motion any more than do the intermolecular forces in the horse and cart themselves.

Example.—A mass of 15 lb. is pulled along a horizontal table by a light inextensible string passing over a smooth pulley and carrying a mass of 1 lb. Find the tension (T) in the string, and the acceleration (*a*) of the system.

Consider the forces acting on the 15 lb. mass. They are :—

(i) In a vertical direction the weight (15×32 poundals) acting vertically downwards which is balanced by the upward reaction of the table.

(ii) In a horizontal direction there is the tension T (poundals). Hence

$$T = 15a.$$

Now consider the forces on the 1 lb. mass.

(i) There are no forces in a horizontal direction.

(ii) Vertically, the weight (1×32 poundals) acts downwards, and the tension T acts upwards.

The resultant force downwards is

$$32 - T$$

and since this acts on a mass of 1 lb., we have

$$32 - T = 1 \times a.$$

Solving these equations

$$a = 2 \text{ ft.sec.}^{-2} \text{ and } T = 30 \text{ poundals.}$$

Example.—Two masses, m_1 and m_2 ($m_1 > m_2$), are connected by a light string passing over a smooth pulley. Discuss the motion and find the tension, T, in the string.

Let a be the acceleration; consider the mass m_1. Resolving vertically, the downward force is $m_1g - T$, and this acts on a mass m_1. Similarly, for the mass m_2, the upward force is $T - m_2g$.

Hence

$$a = \frac{m_1g - T}{m_1} \text{ or } \frac{T - m_2g}{m_2}.$$

$$\therefore\ a = \frac{(m_1 - m_2)g}{m_1 + m_2}, \text{ and } T = \frac{2m_1m_2}{m_1 + m_2}g.$$

Example. A cage of mass 0·5 ton is drawn up a mine shaft by a rope passing over a smooth pulley at the top. The pull is constant and the cage moves through a distance of 30 ft. in 6 sec. from rest. If the mass of the men inside the cage is 1·5 tons calculate the tension, T, in the rope and the thrust, F, on the floor of the cage.

$$\text{Acceleration} = \frac{2s}{t^2} = \frac{2 \times 30}{36} = \frac{5}{3} \text{ ft.sec.}^{-2}$$

The upward pull on the cage, if T is expressed in ton.-wt., is

$$(T - 2)g = 2 \cdot \tfrac{5}{3}.$$

$$\therefore\ T = 2{\cdot}10 \text{ ton.-wt.}$$

Consider now the vertical forces on the men; there is F, the upthrust of the floor on them, and their weight $1{\cdot}5g$ downwards. Hence

$$(F - 1\tfrac{1}{2})g = \tfrac{3}{2} \cdot \tfrac{5}{3} = \tfrac{5}{2}.$$

$$\therefore\ F = 1{\cdot}58 \text{ ton.-wt.}$$

[F is expressed in ton.-wt.]

Atwood's Machine.—If an attempt be made to determine the acceleration due to gravity by observing the motion of a freely falling body, one will soon realize that the method is not susceptible of any great precision since the time of fall is so short that it cannot be measured directly and accurately. The time of fall may be increased by diminishing the downward forces acting on the body. Atwood's machine, in which this principle is involved, is shown in Fig. 2·6 (a). It consists of two columns, AB and CD, supported on a common base fitted with levelling screws so that the instrument may be made vertical. The upper ends of these pillars are joined together by a piece of wood upon which is supported a pulley wheel E carried by four other wheels—in this way the friction is reduced to a minimum. Two equal masses, P and Q, usually in the form of cylinders, are attached to a cord passing over E. The mass Q carries a rider of the shape shown at (b). Initially Q and its rider rest on a drop bridge L maintained in a horizontal position. When this bridge is lowered the driving force acting upon the system is mg, where m is the mass of the rider and g is the acceleration due to gravity. The load moving is $(2M + m + \kappa)$ where M is the mass of each cylinder and κ is a term added to represent the

inertia of the pulley system. If the whole moves with acceleration a, then

$$mg = (2M + m + \kappa)a$$

or

$$(2M + m + \kappa) = \frac{mg}{a}.$$

In order to eliminate κ the experiment may be repeated using two other equal cylinders of mass M′ when

$$(2M' + m + \kappa) = \frac{mg}{a'}.$$

Subtracting these two equations we have

$$2(M - M') = mg\left[\frac{1}{a} - \frac{1}{a'}\right].$$

From this equation g can be calculated when a and a' are known. To determine the acceleration of a body it is necessary to measure the velocity at two different times. In the present example the body moves from rest, so that its initial velocity is zero. In order to find its velocity, v, at some other time, let us say when the body has moved to R, a ring is placed at this point so that the rider is automatically removed when it reaches this point, and the system continues to move with constant velocity. This is found by observing the time required for the system to move to the lower platform T. If s is the distance RT and the observed time t, the velocity is $\frac{s}{t}$. If we assume that the acceleration over the distance LR is uniform

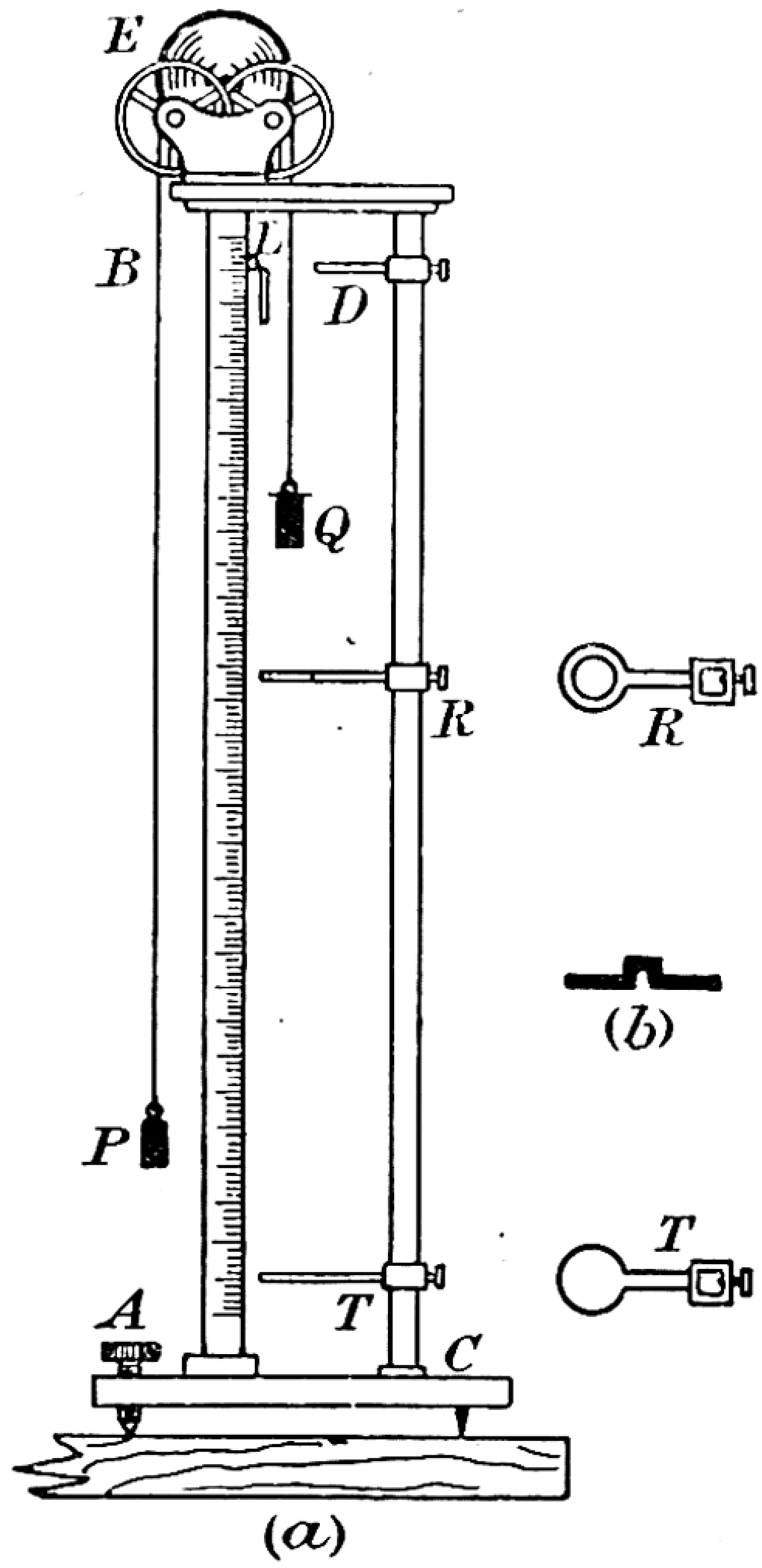


Fig. 2·6.—Atwood's Machine.

$$v^2 = 2ax$$

where x = LR.

$$\therefore a = \frac{v^2}{2x} = \frac{s^2}{2xt^2}.$$

A Modern Form of Atwood's Machine.—CUSSON and JOHNSON have recently designed a form of Atwood's machine in which the method of timing has been considerably improved. The two masses are joined together by a piece of paper in the form of a very light ribbon, Fig. 2·7. This moves past the end of a vibrating arm making a definite number of vibrations per second. The experiment is carried out in exactly the same manner as before, except that the acceleration is now deduced from the wavy trace recorded on the ribbon. An example of such a trace is given in Fig. 2·8. It will be noticed that the initial part of the curve is not very distinct, but by proceeding in the following manner this portion of the curve can be neglected. Imagine a straight line drawn parallel to the edge of the ribbon so that it passes down the centre of the trace. Let A, B, and C be three points at which the wavy line is cut by the straight one and such that the same number of vibrations has been made in the two intervals AB and BC. Let this number be n. If the velocity of the ribbon at A was v_0 and the time taken to make n complete waves t, then

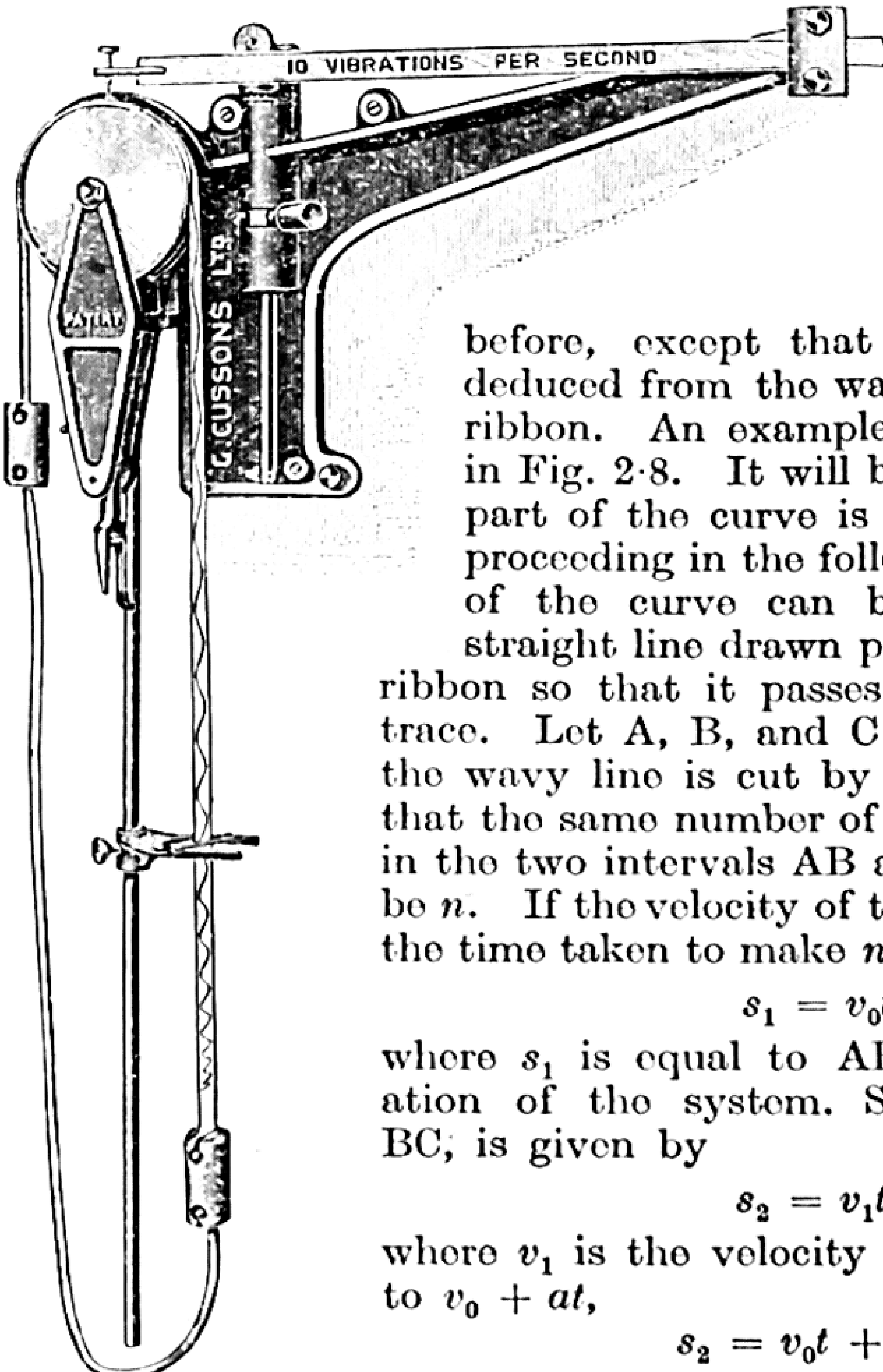


FIG. 2·7.—Modern Form of Atwood's Machine.

$$s_1 = v_0t + \tfrac{1}{2}at^2$$

where s_1 is equal to AB, and a is the acceleration of the system. Similarly s_2, the distance BC, is given by

$$s_2 = v_1t + \tfrac{1}{2}at^2,$$

where v_1 is the velocity at B. Since this is equal to $v_0 + at$,

$$s_2 = v_0t + at^2 + \tfrac{1}{2}at^2.$$

By subtraction we have

$$s_2 - s_1 = at^2,$$

so that a may be calculated. Care must be taken to see that no portion of the wavy curve used corresponds to the time after the rider has been removed, for the acceleration is then zero. But this portion of the curve may be used to demonstrate that the velocity is then constant.

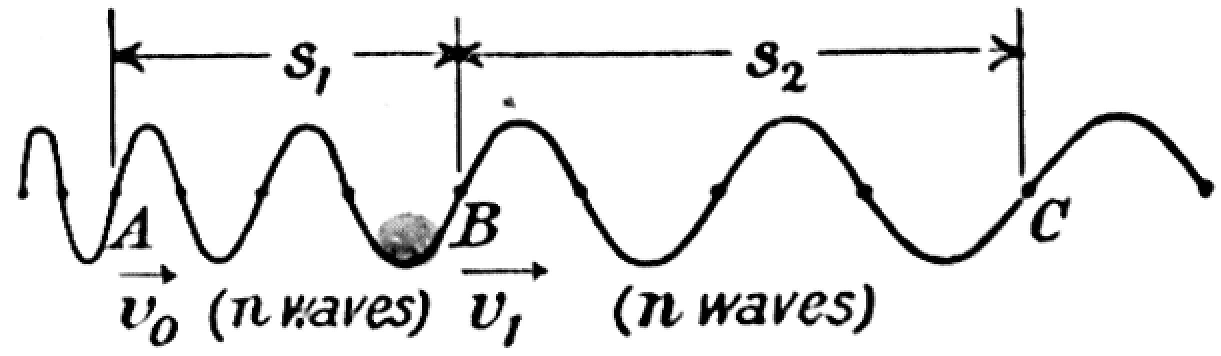


FIG. 2·8.

Welander's Apparatus for Determining 'g.'—A long pendulum, Fig. 2·9, consisting of a thin steel wire and an iron ball B_1 about two inches in diameter, is suspended from the ceiling or wall bracket.

The pendulum is held at an anglo to the vertical by an electromagnet, M_1. Another electromagnet, M_2, wired in series with the first, holds a release ball B_2, also of steel. By opening the switch, S, the pendulum and ball B_2 are released simultaneously. On falling to a vertical position the pendulum operates a gate switch, G. A trap switch, T, operates when struck by the falling ball.

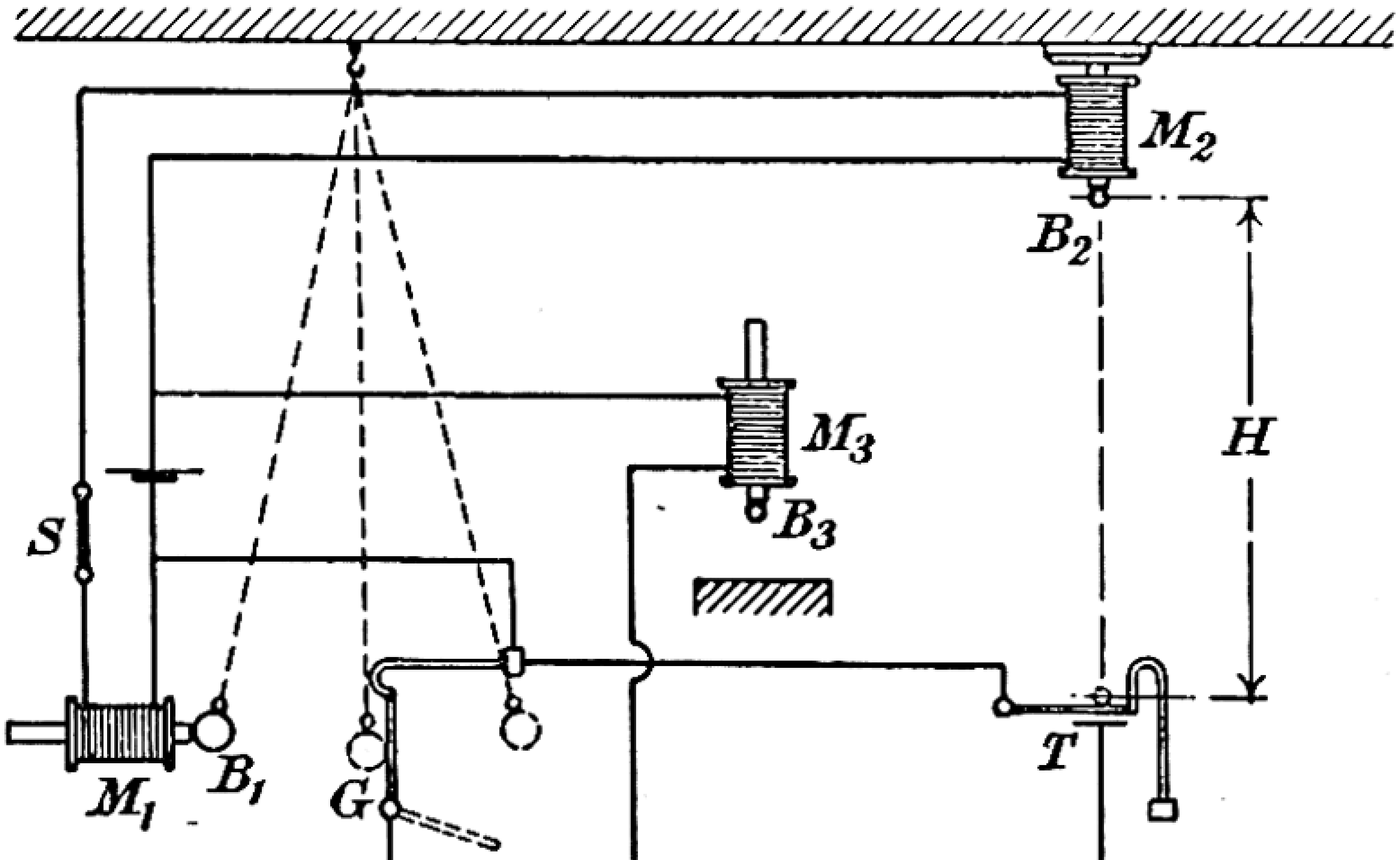


FIG. 2·9.—Welander's Apparatus for Determining 'g.'

To begin the experiment the gate and trap switches are set and the switch S closed. The three electromagnets are now excited so that the pendulum and two spheres may be fixed in position. The trap having been placed at a distance H below B_2, the switch S is opened when the pendulum and ball B_2 are set free together. Let us suppose that the ball B_2 hits the trap switch and closes it before the pendulum strikes the gate G. The electromagnet M_3 will still be excited and B_3 will remain in position. If, however, the gate G is opened before the trap T is closed by impact of the falling sphere, the magnet M_3 is no longer excited and B_3 is released. The trap is therefore moved an inch at a time until two positions of the trap are found such that at one the ball B_3 remains in position while at the other B_3 is set free. By moving the trap 0·1 inch at a time two positions are determined, the lower of which releases the indicator ball while the higher leaves it attached.

Let H be the height through which the ball B_2 falls, and t the periodic time of the simple pendulum used [this must be measured as explained on p. 38]. The release ball strikes the trap after a time $\frac{t}{4}$ so that its velocity is $\frac{gt}{4}$ cm. sec.$^{-1}$ if c.g.s. units are used. But this is equal to $\sqrt{2g\mathrm{H}}$ cm. sec.$^{-1}$. We therefore have

$$\sqrt{2g\mathrm{H}} = \frac{tg}{4}, \text{ or } g = \frac{32\mathrm{H}}{t^2} \text{ cm. sec.}^{-2}.$$

Work.—If a constant force F acts on a body and the point of application of the force moves a distance s along the line of action of the force, work is said to have been done on the body. It is given by $W = Fs$.

Units of Work.—When a force of 1 poundal is applied so that its point of application moves 1 ft. along the direction in which the force is acting, the work done is ***1 ft.-poundal.*** Engineers use a larger unit called the ***ft. lb.-wt,***[1] which is the work done under conditions similar to the above when the force is 1 lb.-wt.

In the c.g.s. system the absolute unit of work is the ***erg,*** and this is defined as the ***work done when the point of application of a force of 1 dyne moves 1 cm. along its line of action.*** The practical unit is the joule, or 10^7 ergs.

Energy.—If a body is capable of doing work it is said to possess ***energy,*** i.e. the energy of a body is a measure of its capacity for doing work. An agent performing 550 ft. lb.-wt. of work per second does work at a rate of one ***horse-power.*** In the c.g.s. system the power, or rate at which work is done, is often measured in watts, a watt being equivalent to 10^7 erg. sec.$^{-1}$. Electrical engineers find this unit too small for practical purposes so that they generally employ as their unit one which is equal to a thousand watts; it is termed a kilowatt. One horse-power is 0·746 kilowatt. Electrical energy is measured in Board of Trade units, one of which is equal to one kilowatt-hour.

When a mass m is at rest at a height h, it is attracted to the earth by a force mg [its weight]. If it falls to the earth's surface it does work mgh. But $2gh = v^2$, where v is the final velocity of the body, so that the work done is $\frac{1}{2}mv^2$. The body possessed energy when at rest, for it was capable of doing work equal in amount to $\frac{1}{2}mv^2$. The energy which a body possesses in virtue of its position is called its ***potential energy and is measured by the amount of work the body performs in passing from its original position to a standard position, where the potential energy is considered to be zero.*** The potential energy of a body at the earth's surface is taken as zero, whilst the energy associated with its motion is called its ***kinetic energy.***

$$\therefore \text{Energy} = mgh = \tfrac{1}{2}mv^2.$$

Although the expression $\frac{1}{2}mv^2$ has been obtained for motion under gravity, it is true in general as an expression for the kinetic energy of a mass m moving with velocity v.

Principle of the Conservation of Energy.—From the above it is seen that potential and kinetic energy are mutually convertible—a property which is possessed by all forms of energy, whether they be magnetic, electric, kinetic, etc. When such changes of energy take place ***the principle of the conservation of energy states that the total energy in any system is always constant.***

[1] This is often called a foot-pound, but the notation adopted here leads to less confusion.

Motion in a Horizontal Circle.—If a particle of mass m is moving in a horizontal circle of radius r, with uniform speed v, its velocity cannot be said to be uniform because its direction is continuously changing, and this implies the existence of a force which, in turn, gives rise to an acceleration which can be calculated as follows. Let A and B, Fig. 2·10, be two positions of a particle of mass m rotating in a circle, centre O and radius r. Let v be the speed of the particle, and let us assume that the time required for the body to move from A to B is small. The velocity at A is v and is directed along the tangent AT. At B the velocity is v along BS. If θ is the $\widehat{AOB}$, the velocity at B is equivalent to a velocity $v \sin \theta$ along BD, where BD is parallel to AO, together with a velocity $v \cos \theta$ along BC, where BC is parallel to the tangent AT. If θ is small these are respectively $v\theta$ and v. At A the component velocity along AO was zero so that the change in velocity in this direction is $v\theta$ and this has occurred in a time $\frac{AB}{v}$ or $\frac{r\theta}{v}$. The acceleration is therefore $v\theta \div \frac{r\theta}{v}$, i.e. $\frac{v^2}{r}$. Since the velocity in the direction of the tangent at A does not change by a finite amount, the acceleration along the tangent is zero. The only force acting on the particle is therefore $\frac{mv^2}{r}$ and this is directed towards the centre O; it is called the ***centripetal force***. This force is due to the action of some other body, and since, according to Newton's Third Law of Motion, action and reaction are equal and opposite, it follows that this other body is being pulled by a force which tends to move it from the centre of the circle. This latter is the ***centrifugal force***. Thus, when a stone, attached to one end of a string, is caused to rotate, the pull on the hand of the person performing this experiment is the centrifugal force. The existence of this centrifugal force may also be demonstrated in the following manner. A small container, partly filled with water, is suspended from a string and caused to rotate rapidly in a vertical circle. No water is lost because the velocity is so great that the water exerts a thrust on the bottom of the container which is greater than the weight of the water.

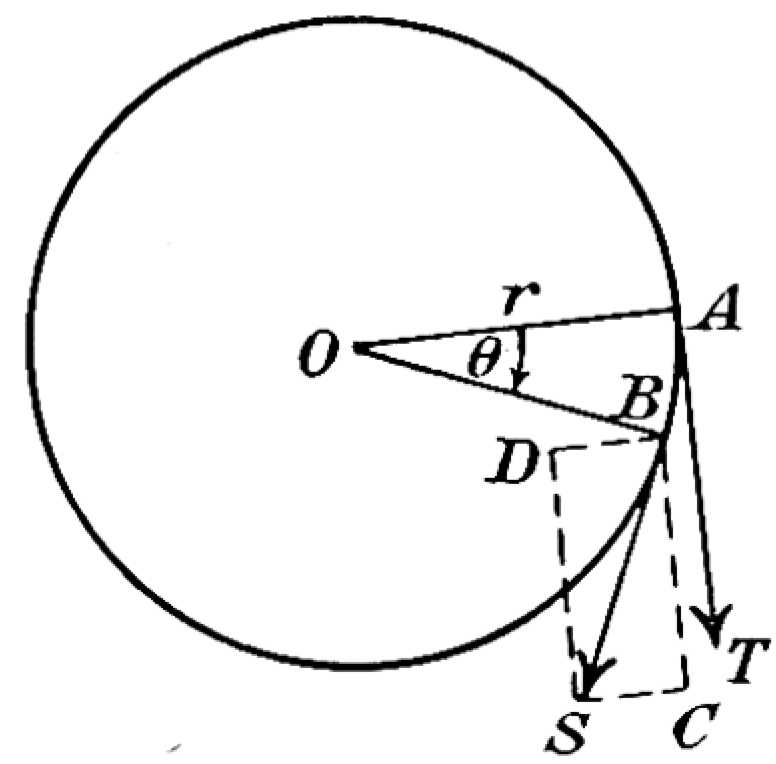


FIG. 2·10.—Motion in a Horizontal Circle.

In chemical laboratories centrifugal force is utilized in the separation of small crystals from the mother-liquors. Dairy farmers also use this force when they separate cream from milk by mechanical means, and in the purification of honey. Dyers are in the habit of rotating their yarns in this way so that they may lose their moisture more readily. In preparing flake nickel for use in Edison storage batteries, the flake is washed and then centrifuged in order to remove the water.

The Banking of Tracks.—Fig. 2·11 represents a truck of mass m moving with speed v on a circular track banked at an angle θ. If G is the centre of gravity of the truck, mg will act vertically downwards along GA whilst the centrifugal force $\frac{mv^2}{r}$ will be operative in a horizontal plane along GB. To prevent the truck from leaving the rails the track must be banked to such an extent that the resultant of these two forces is perpendicular to the track. Under these conditions we have

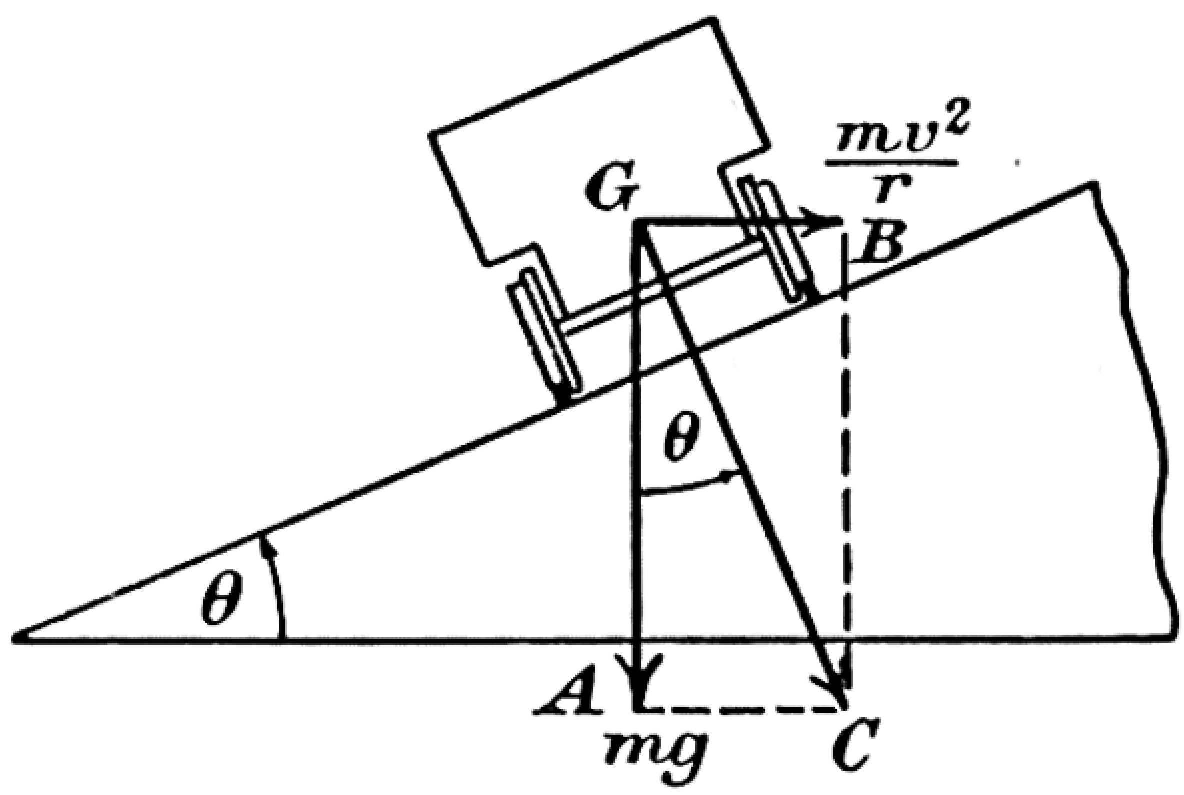


FIG. 2·11.—Truck on a Banked Track.

$$\tan \theta = \frac{\mathrm{AC}}{\mathrm{GA}} = \frac{mv^2}{r} \div mg = \frac{v^2}{rg},$$

where θ is the angle of greatest tilt on the track surface.

If the outer rail is not thus elevated the flanges of the wheels will grind against it in order to create a force sufficient to enable the truck to take the curve at the desired speed.

Simple Harmonic Motion.—Let us imagine that a point is moving with uniform angular velocity ω along the circumference of a circle whose centre is O—cf. Fig. 2·12. If PM is drawn perpendicular to the diameter AOC, M will move to and fro across this diameter as P moves round the circle. The point M is said to execute simple harmonic motion, so that we may say that simple harmonic motion [S.H.M.] is the projection of uniform

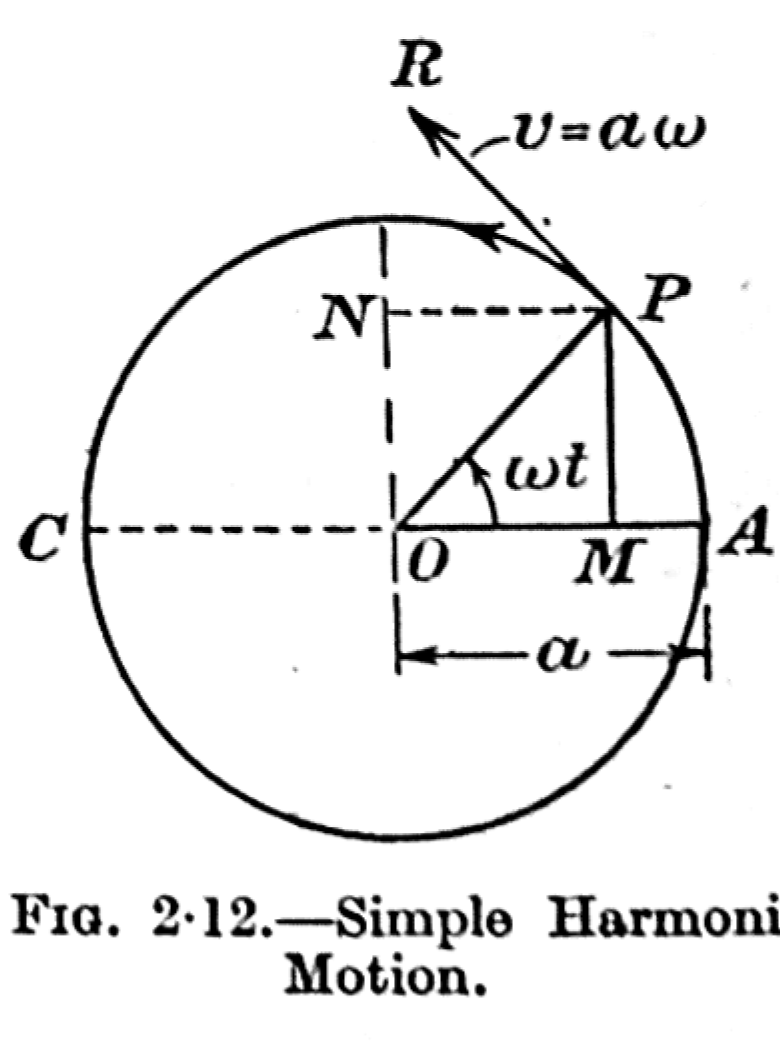


FIG. 2·12.—Simple Harmonic Motion.

motion in a circle upon a diameter of the circle. The distance OA is called the amplitude of the oscillation and the time, T, required for one complete oscillation, i.e. for the point M to move from A to C and back again, is referred to as the period of the oscillation. It is given by the equation $\omega T = 2\pi$, for T is the time required for the moving particle to rotate through an angle 2π.

Let time be reckoned from the instant when the moving point is at A. At time t the moving particle will have reached a point P where $\widehat{AOP} = \omega t$.

Now the velocity of M is equal to the velocity of the point P projected upon AOC, i.e. $a\omega \times \cos \widehat{RPN}$, where $PR = v = a\omega$, and is tangential to the circle at P, and PN is parallel to AO.

$$\therefore v_M = a\omega \sin \omega t. \qquad \because \widehat{RPN} = 90° - \omega t.$$

These equations show the velocity of M is zero at A and C, and reaches a maximum value $a\omega$ at O; the acceleration, on the other hand, is a maximum when M is at A or C, i.e. at either extremity of its path, but it is zero when M is at O [cf. below].

Formulæ for Simple Harmonic Motion.—Let a be the amplitude, while ω is the angular velocity of the point P. The actual speed of P is therefore $a\omega$ so that its acceleration is $a\omega^2$ in the direction PO. The acceleration of M is equal to the component of the acceleration of P parallel to AO, viz. $a\omega^2 \cos \widehat{POM}$. But since $\cos \widehat{POM} = \dfrac{OM}{OP}$, this reduces to $\omega^2 . OM$. We therefore see that the acceleration of M is directly proportional to its distance from O since ω^2 is a constant. When a body moves so that its acceleration is always proportional to its distance from some fixed point in the line of motion and directed towards that point, its motion is said to be simple harmonic.

The periodic time, T, is equal to the time occupied by P moving once round the circle. Now in this time P moves with a speed v a distance $2\pi a$, so that $T = \dfrac{2\pi a}{v} = \dfrac{2\pi}{\omega}$.

Since ω^2 is the acceleration when the particle is at unit distance from the origin, we have

$$T = 2\pi/\sqrt{\text{acceleration for unit displacement}} = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}.$$

To Determine the Period of a Simple Pendulum.—A simple pendulum is really a mathematical ideal to which we can only approximate in practice, for it is defined as a heavy particle suspended from a rigid support by a massless inextensible string. The pendulum we have to use consists of a heavy 'bob' suspended by

a light cord. Let A, Fig. 2·13 (*a*), be the 'bob' of mass m, C the point of suspension, so that AC is the string of length l. Let OC be the rest position of the pendulum. Let θ be the $\widehat{ACO}$. Then when the mass is at A the force acting upon it in the direction of the tangent is $mg \sin \theta$, which we may replace by $mg\theta$ if θ is small. The acceleration of A is therefore $g\theta$, i.e. $g \cdot \frac{OA}{l}$, or $\left(\frac{g}{l}\right) \times$ arc OA. Hence, when θ is small, the acceleration is proportional to the dis-

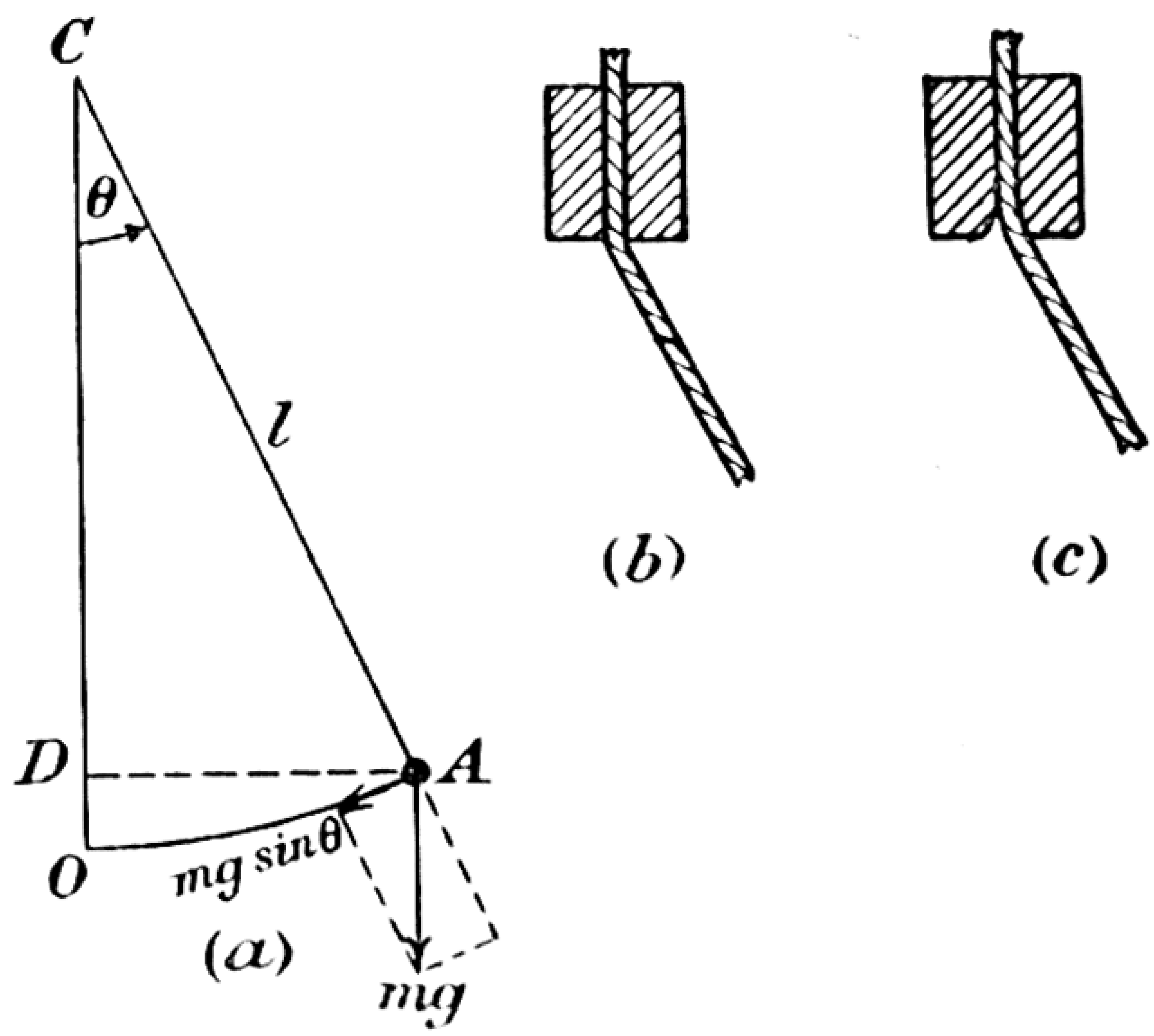


FIG. 2·13.—Simple Pendulum.

tance of A from O so that A will execute a simple harmonic motion, the period of which is

$$T = 2\pi \sqrt{\frac{l}{g}}.$$

To Determine the Acceleration due to Gravity.—The simple pendulum furnishes us with a ready means of finding 'g.' A lead sphere about one inch in diameter is suspended by a thread held between two blocks as shown at (*b*) so that the length of the pendulum does not vary as it swings. If the lower edges of these blocks are not in the same plane or have become rounded as in (*c*), considerable errors will be introduced. This pendulum is placed in front of some vertical line to indicate the rest position, or it may be viewed with a telescope which is adjusted so that the vertical cross-wire in the telescope coincides with the zero position of the extremity of the pendulum. The pendulum is set in motion and the time of twenty complete oscillations found. The observations are repeated and the mean time calculated. To determine this more accurately the time is noted when the pendulum swings

past its zero position. The swings are ***not*** counted, but after the lapse of some minutes, or previously if the motion is damped considerably, the time is again observed when the pendulum moves through its zero position. The time between these two 'coincidences' divided by the approximate value of the periodic time would be an integer if the observations were not subject to error. The integer nearest to the quotient actually obtained is therefore the number of swings made between the two coincidences; since this is known the period can be calculated. This method of measuring a period should always be resorted to whenever possible.

The length of the pendulum is then changed and the period determined. In this way a series of corresponding values connecting T and l are obtained. Since l is equal to $\frac{gT^2}{4\pi^2}$ it follows that if T^2 [abscissa] is plotted against l [ordinate] the slope of the line will be $\frac{g}{4\pi^2}$; if this is measured g may be deduced [cf. p. 14].

In evaluating the 'slope' of a straight line it must be understood quite clearly that it is not the tangent of the actual angle which the straight line makes with the axis of x, for this depends on the scales used to plot the variables. If A and B are two points on a straight line, and C the point of intersection of straight lines through A and B parallel to the axes Ox and Oy respectively, then the slope of the line is equal to

$$\frac{\text{the quantity represented by CB}}{\text{the quantity represented by AC}}.$$

Motion of a Liquid in a U-tube.—Let l be the total length of liquid [say mercury] contained in a **U**-tube of uniform cross-section. Let m be the mass of liquid per unit length of the tube. If the mercury is depressed in one limb so that it rises an equal distance in the other, on releasing the mercury it will continue to oscillate with a definite period which may be calculated as follows. When the mercury in one limb is at a height x above its zero position the force operative on all the mercury in the tube is equal to the weight of a mercury column of length $2x$, i.e. $W = 2mxg$. Since the total mass of mercury is ml the acceleration, a, at this particular instant is given by

$$2mxg = mla, \text{ or } a = \frac{2gx}{l}.$$

We therefore see that the acceleration is proportional to the displacement x; i.e. the motion is simple harmonic and the periodic time is

$$T = 2\pi\sqrt{\frac{l}{2g}},$$

since $\frac{2g}{l}$ is the acceleration for unit displacement, i.e. the periodic time is the same as that of a simple pendulum of length $\frac{l}{2}$.

Motion of a Body Suspended by a Spring.—We now consider the motion of a heavy body, suspended from a fixed support by a helical spring of negligible mass. Let M be the mass of the body. When the spring is at rest its lower end will be at some definite position. When a small additional mass m is added, let x be the distance through which the lower end of the spring descends. Experiment shows that if a mass m had been removed from the total load carried by the spring the equilibrium position of the lower end of the spring would have been raised by an amount x. It also shows that x is directly proportional to m.

If therefore the load is M and the spring is stretched further by an amount y, the force tending to restore the load to its equilibrium position is the resultant of the weight Mg acting vertically downwards and the upward pull $\left(M + \frac{m}{x}y\right)g$, viz. $\frac{mgy}{x}$. The acceleration, a, of the mass M will therefore be given by $a = \frac{mg}{Mx}y$, i.e. it is proportional to the displacement y. The motion is therefore simple harmonic with a period T given by

$$T = \frac{2\pi}{\sqrt{(\text{acceleration for unit displacement})}}$$
$$= 2\pi\sqrt{\frac{Mx}{mg}}.$$

The Theory of Dimensions.

The Dimensions of a Physical Quantity.—It has already been seen that the magnitude of a physical quantity may be expressed in terms of an appropriate unit, i.e. a given quantity is said to be so many times a certain unit. The statement that the length of a particular wall is a metres implies that its length is a times a certain unit of length—the metre. The above statement really consists of two parts—

(i) the pure number or numeric a which is the measure of the quantity in terms of the unit employed,

(ii) the name of the unit.

Now the measure of a physical quantity varies according to the size of the unit employed, but the product of the measure of a physical quantity and the unit employed remains constant. Thus,

$$2 \text{ metre.} = 200 \text{ centimetre.}$$

If n and n_1 are the measures of a particular physical quantity when the units are $[U]$ and $[U_1]$ respectively, then

$$n[U] = n_1[U_1],$$

i.e.

$$n \propto \frac{1}{[U]},$$

or the measure of a physical quantity is inversely proportional to the size of the unit in which that physical quantity is expressed.

In selecting the units of length, mass, and time the choice is arbitrary. When we have to deal with velocity for example, however, we could still choose a certain velocity as the unit velocity. The unit chosen must satisfy the following requirements:—It must be reproducible and capable of being easily applied. Such a unit of velocity is not easy to find, but the difficulty is overcome in the following way. Suppose that the engine known as the 'Royal Scot,' when travelling at its maximum speed, takes a seconds to pass from one end of the platform of a certain station to the other, the distance between these ends being b cm. The velocity of the engine may then be said to be unity. According to this scheme, the velocity of an object moving 1 cm. in a sec. would be $\frac{1}{b} \times$ (the above unit of velocity). If the distance moved were 1 cm. in 1 sec., the velocity would be $\frac{a}{b}$ units. If an object travelled l cm. in t secs. its velocity in terms of the unit selected would be $\frac{a}{b} \cdot \frac{l}{t}$. In all such expressions the factor $\frac{a}{b}$ occurs. Why not get rid of it by choosing a more suitable unit? Let the unit of velocity be such that an object moving with unit velocity travels 1 cm. in 1 sec. Then the velocity of a body describing l cm. in t secs. is $\frac{l}{t}$ cm.sec.$^{-1}$. The unit velocity is therefore expressed in terms of the units of length and of time. Such a unit is known as a ***derived unit.***

The unit of velocity thus selected is directly proportional to the unit of length and inversely proportional to the unit of time, for if a body moves 1 metre in a second its velocity will be 100 times that of a body moving with unit velocity, while if it moves 1 cm. in 1 minute, its velocity will be 60 times less. We say that the dimensions of the unit of velocity are 1 in length and -1 in time, a fact represented symbolically as $[L][T^{-1}] = [V]$.

Dimensional Equations.—The interrelationship between the units of length, mass, and time—the so-called ***absolute units***—and a derived unit may be expressed by means of a dimensional equation, where by the statement that the dimensions of a certain

physical quantity are α, β, and γ in length, mass, and time respectively, we mean that the unit in terms of which the quantity is expressed varies as

$$[L]^{\alpha}\ [M]^{\beta}\ [T]^{\gamma},$$

where $[L]$ denotes the unit of length, $[M]$ that of mass, and $[T]$ that of time.

An expression such as $n[L]$ implies that the length of a certain object is n times the unit of length. n is itself a mere number. To discover the manner in which the unit of area depends on that of length let us consider the area of a rectangle whose adjacent sides are b and c. Then its area, a, is

$$\begin{aligned} a[A] &= b[L] \times c[L] \\ &= bc[L]^2. \end{aligned}$$

This equation shows that if, for example, the unit of length is doubled, that of area is quadrupled, i.e. the number expressing the area in terms of the new unit will be reduced to one-fourth of the number expressing the area in terms of the old unit.

Let us consider the dimensions of the unit of density in respect to the dimensions of the three fundamental units. We have

$$\text{density} = \text{mass/volume}.$$

Therefore

$$[D] = [M]/[L]^3 = [M]\,[L]^{-3}.$$

The dimensions of density are therefore one in respect to the unit of mass and -3 in respect to that of length.

Such equations as these are useful in two ways :—

(i) we can express a density given in one system of units in terms of any other possible system of units,

(ii) in any equation in which a number of terms are added together the different terms must be homogeneous as far as their dimensions are concerned, i.e. each term must be expressed in the same units of dimensions. This fact was first pointed out by Fourier, a celebrated French mathematician.

The Dimensions of Some Physical Quantities in Mechanics.—In determining the dimensions of a unit in which a physical quantity may be expressed, it is only necessary to write down any equation, so long as it is valid, connecting this quantity with others whose dimensions are known. The particular equation may be either one applicable to a particular instance or one that is true in general.

(i) *Velocity.* We have

$$s = vt$$

where s is a numeric representing the distance traversed in t seconds

by a body moving with constant velocity v, [t and v are numerics]. Then

$$s[L] = v[V] \, . \, t[T]$$

where $[V]$ denotes the dimensions of the unit in which a velocity is expressed. From the above it follows at once that

$$[V] = [L]\,[T]^{-1}.$$

(ii) *Acceleration.* We have $s = \frac{1}{2}at^2$, so that

$$s[L] = \tfrac{1}{2}a[A] \, . \, t^2[T]^2$$

where $[A]$ denotes the dimensions of the unit in which an acceleration may be expressed. Hence

$$[A] = [L]\,[T]^{-2}$$

(iii) *Momentum* (I). We have

$$I = mv,$$

so that

$$I[I] = m[M] \, . \, v[LT^{-1}]$$

$$\therefore\ [I] = [M][L][T]^{-1}.$$

(iv) *Force* (F). We have $Ft = I$.

$$\therefore\ [F]\,[T] = [MLT^{-1}]$$

$$\therefore\ [F] = [M][L][T]^{-2}.$$

(v) *Work* (W). $W = F \, . \, s$.

$$\therefore\ [W] = [MLT^{-2}][L]$$
$$= [M][L]^2[T]^{-2}.$$

(vi) *Power* (P). $Pt = W$.

$$\therefore\ [P] = [M]\,[L]^2[T]^{-3}.$$

Some Applications of the Method of Dimensions.—

(i) *The Period of a Simple Pendulum.* This may depend on

(*a*) the mass, m, of the bob,
(*b*) the length, l, of the string,
(*c*) the acceleration, g, due to gravity.

Let us suppose that $t = \kappa m^{\alpha} l^{\beta} g^{\gamma}$, where α, β and γ are the 'exponents of the dimensions,' and κ is the constant.

$$\therefore\ [T] = [M]^{\alpha} \, . \, [L]^{\beta} \, . \, [LT^{-2}]^{\gamma}.$$

Equating like exponents, we have

$$\alpha = 0,\ \beta + \gamma = 0,\ 1 = -2\gamma.$$

$$\therefore\ \gamma = -\tfrac{1}{2},\ \beta = \tfrac{1}{2}.$$

$$\therefore\ t = \kappa \, . \, \sqrt{\frac{l}{g}}.$$

The constant κ may be determined experimentally.

(ii) *The Natural Frequency* (f) *of a Uniform Stretched Wire.* This may depend on

(*a*) the mass, m, of the wire,
(*b*) its length, l,
(*c*) the stretching force, F.

If $f = \kappa m^{\alpha} l^{\beta} F^{\gamma}$, where α, β, and γ are the 'exponents of the dimensions,' and κ is a constant.

$$\therefore [T^{-1}] = [M]^{\alpha}[L]^{\beta}[MLT^{-2}]^{\gamma}.$$
$$\therefore 2\gamma = 1, \alpha + \gamma = 0, \beta + \gamma = 0.$$
$$\therefore \alpha = -\tfrac{1}{2}, \beta = -\tfrac{1}{2}, \gamma = \tfrac{1}{2}.$$
$$\therefore f = \kappa\sqrt{\frac{F}{ml}}.$$

If μ = mass per unit length of the wire, $m = \mu l$, and

$$f = \frac{\kappa}{l}\sqrt{\frac{F}{\mu}}.$$

EXAMPLES II

1.—A body, initially travelling with a velocity of 10 ft. sec.$^{-1}$, is observed to be moving with a velocity of 13·1, 15·2, and 16·3 ft. sec.$^{-1}$ at the end of the 1st, 2nd, and 3rd seconds of its motion respectively. Determine the distance traversed.

2.—A train is travelling in a curve of 1 mile radius at a rate of 20 m.p.h.; through what angle has it travelled in 15 secs.?

3.—What do you understand by the term acceleration? A particle has an initial velocity of 14 ft. sec.$^{-1}$. After traversing 100 ft. its velocity is 19·5 ft. sec.$^{-1}$. What is the acceleration, and how long has it been moving?

4.—A body starting from rest is observed to traverse 60 cm. in the 8th second of its motion. What is the acceleration?

5.—A body is thrown upwards with a velocity of 153·2 ft. sec.$^{-1}$. What time elapses before it reaches the highest point in its motion? How high does it rise?

6.—Two buckets, each of mass 7·5 lb., are supported by a thin rope over a smooth pulley and are at rest. A mass of 1 lb. is dropped from a height of 4 ft. into one of the buckets. Calculate the time which elapses before the system has moved through a vertical distance of 10 ft.

7.—Explain what information may be obtained from a graph in which velocity is plotted against time. A train starting from rest accelerates uniformly until it has traversed $1\frac{1}{2}$ miles; its speed then remains constant for the next $2\frac{1}{4}$ miles when an application of the brakes produces a uniform retardation bringing it to rest after a further $\frac{3}{4}$ mile. If the whole journey occupies $7\frac{1}{2}$ minutes find the maximum speed in miles per hour. (L.S.C.)

8.—A ball is thrown at an angle of 45° to the horizontal so that at the top of its flight it enters a window 36 ft. above the thrower. Find the speed at which it was thrown and the distance of the wall containing the window from the thrower. (L.S.C.)

9.—Distinguish between *momentum* and *kinetic energy*. Which is conserved during a collision and what happens to the other? A

bullet weighing 30 gm. and travelling at 500 metre. sec.$^{-1}$ embeds itself in a suspended lump of wood of mass 7·47 kilograms. How far will this block have risen above its original position when it reaches the end of its swing? If the length of the suspension is 50 cm. how far will the block have swung in a horizontal direction? (Take: g 1,000 cm.sec.$^{-2}$.] (L.S.C.)

10.—A boy weighing 8 stone and riding a bicycle weighing 21lb. rides up a hill with a gradient of 1 in 21 at 9 ml.hr.$^{-1}$. Assuming that friction is equivalent to a force of 2 lb.-wt. resisting his motion up the hill, find how much work he is doing per second.

11.—Two bodies initially at rest and of mass 10 gm. and 50 gm. respectively are each acted on by a force equal to the weight of a body of mass 4 gm. Compare the times for which these forces must be operative to produce (*a*) the same kinetic energy, (*b*) the same momentum.

12.—Describe the variations of velocity and acceleration of a body moving with simple harmonic motion. If, in a simple harmonic motion, the amplitude of the displacement is 10 cm. and the period 3 seconds, what are the maximum values of the velocity and the acceleration? (B.S.S.C.)

13.—Define two units of force which are in common use. Calculate the force necessary to bring to rest a motor-car weighing 2 tons travelling at a speed of 30 ml.hr.$^{-1}$, in a distance of 20 yds.

14.—Derive an expression for the period of a body moving with simple harmonic motion, in terms of its acceleration and displacement. A vertical U-tube of uniform cross-section contains mercury to a height of 20 cm. If the liquid on one side is depressed, and then released, the mercury oscillates up and down the two sides of the tube. Show that the motion is simple harmonic, and calculate its period.

15.—Define potential energy and kinetic energy, and state the units in which each is measured. A block of wood weighing 500 gm. is allowed to fall down an inclined plane which makes an angle of 30° with the horizontal. After sliding a distance of 20 cm. from rest it is moving with a velocity of 50 cm.sec.$^{-1}$. How much energy has the block lost at this point? What has become of the energy?

16.—Distinguish between *momentum* and *kinetic energy*.

A simple pendulum l metres long has a bob of mass m gm. Derive expressions for the momentum and the kinetic energy of the bob at its lowest point, if the pendulum swings 30° from the vertical.

CHAPTER III

THE ELEMENTS OF STATICS

In the previous chapter it has been shown that whenever a force acts on an object which is not fixed, then that body moves. If the body is to remain at rest it must be acted upon by an equal and opposite force or its equivalent. Under such conditions the body is said to be in ***equilibrium***, and statics is that branch of physics which studies the properties of bodies in equilibrium. The bodies are supposed to be rigid, homogeneous and not too large, for otherwise the lines of action of all the gravitational forces acting on the individual parts of the body would not be parallel to one another and the problem of determining the line of action of the resultant of such a system of forces is, in general, not capable of solution.

Just as a velocity can be represented by a straight line, so can a force be similarly represented, for this latter has magnitude, direction, and sense.

Resultant of Two Non-Collinear Forces.—If OA and OB, Fig. 2·2, p. 20, represent two forces, F_1 and F_2, the resultant, F, is represented by the diagonal OC, since forces are vectors. Its magnitude is given by

$$F^2 = F_1^2 + F_2^2 + 2F_1F_2 \cos \theta,$$

where θ is the angle AOB.

The Experimental Verification of the Law of Forces.—The experimental arrangement is shown in Fig. 3·1. Three spring balances L, M, and N are supported on hooks, and joined together by means of three pieces of string knotted together at O. The readings of the two balances M and N are observed, and these are a measure of the tensions in the strings. Immediately below the strings a piece of paper is attached to the board which supports the apparatus, and upon this paper straight lines are drawn parallel to the strings leading to M and N. Along these lines distances OA and OB respectively are marked off, their lengths being proportional to the readings of M and N respectively. The parallelogram OACB is then completed, and the tension in L should be proportional to the length of OC whilst the directions OL and OC should be parallel.

Now the reading of the spring balance L measures that force which prevents O from moving when acted upon by the forces in M and N, i.e. the force in L is the ***equilibrant*** of these two other

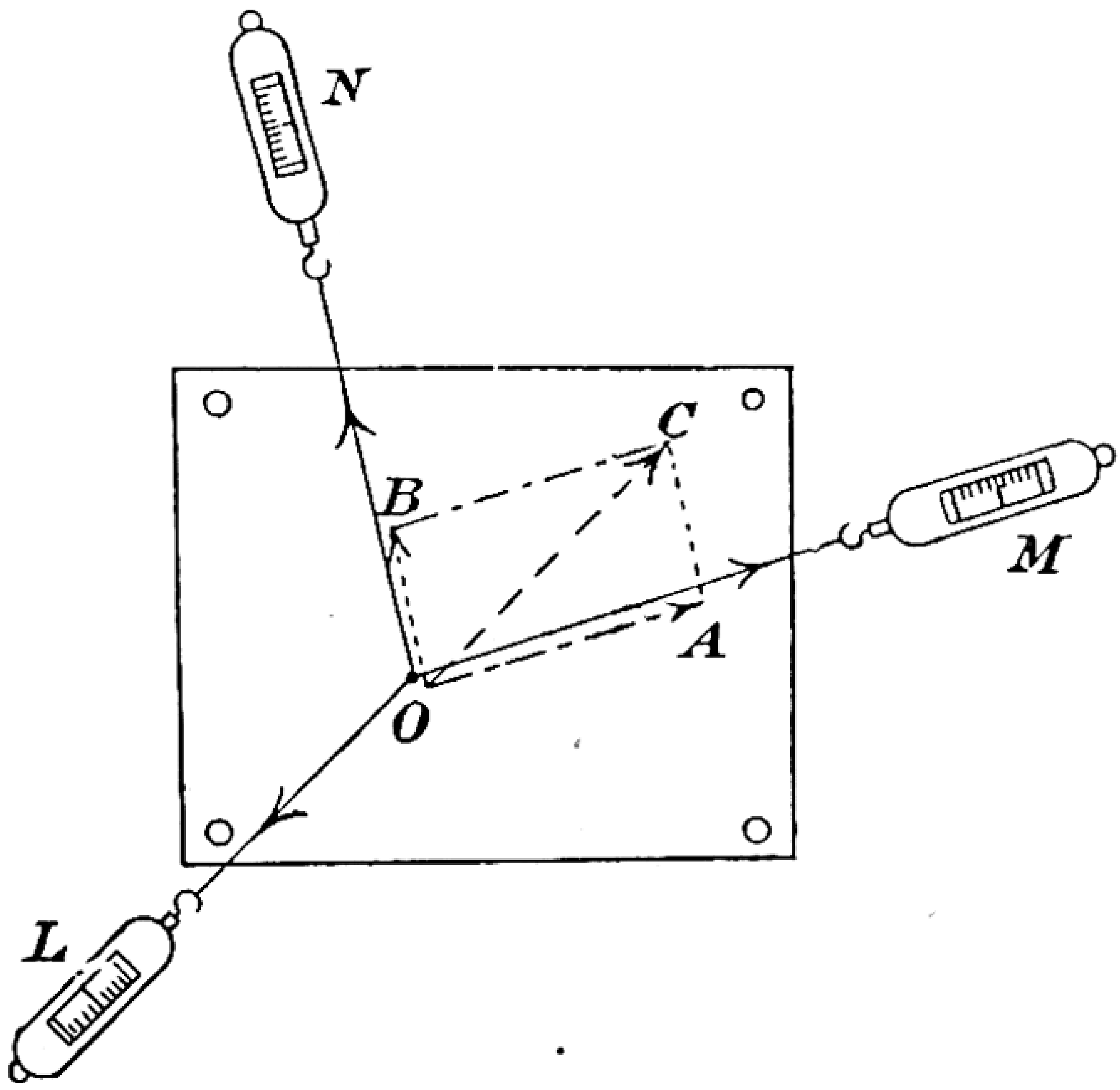


FIG. 3·1.—Verification of the Parallelogram Law of Forces.

forces, whereas the resultant of the forces represented by OA and OB is represented by OC and is equal and opposite to the equilibrant.

Parallel Forces.—When two or more non-collinear parallel forces act upon a rigid body the line of action of the resultant may be found very easily in the following way. If OA and O′A′ Fig. 3·2, represent completely two parallel forces acting on a rigid body, join OO′ and at O and O′ insert two equal and opposite forces OB and O′B′. These will not affect the equilibrium of the body. The two forces at O and O′ are combined according to the parallelogram law, and so we have their resultants OC and O′C′. The lines of action of these two

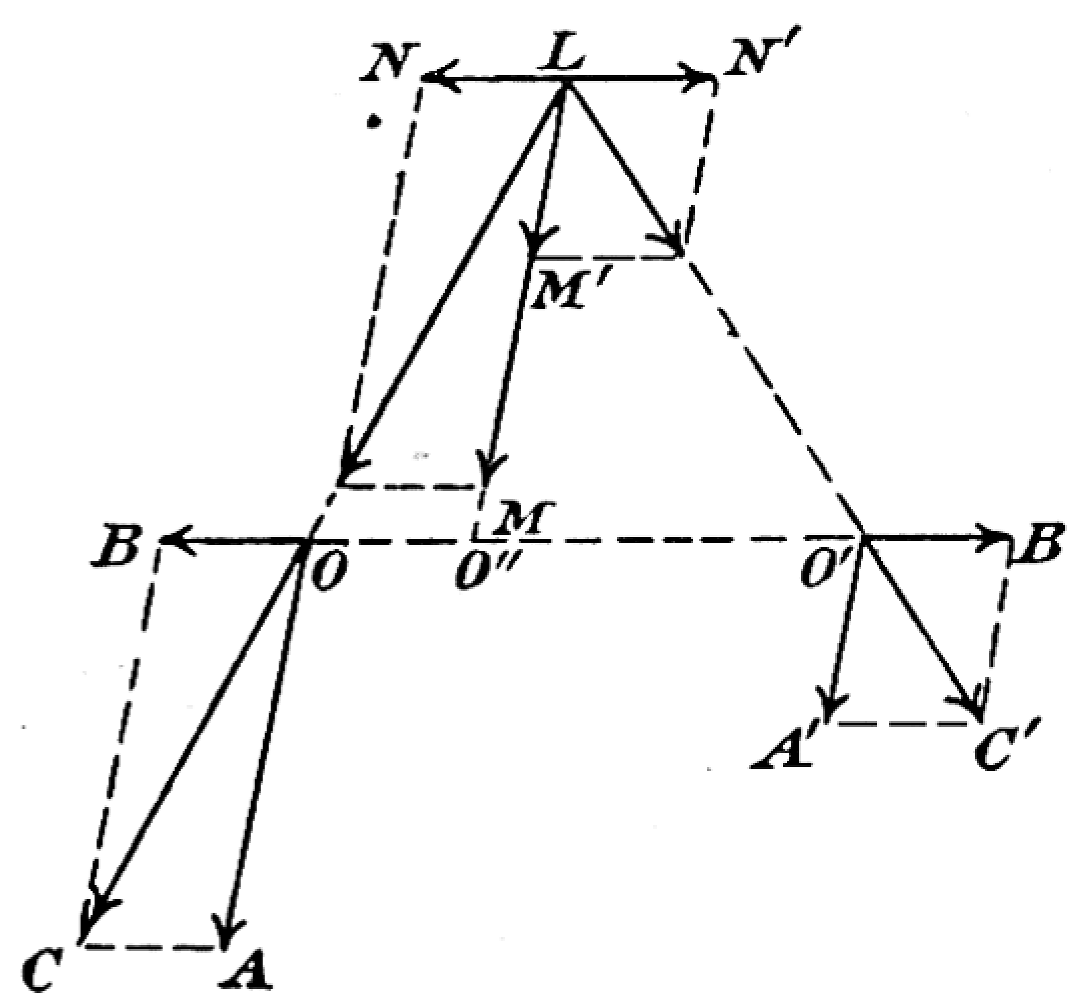


FIG. 3·2.—Graphical Determination of the Resultant of two Parallel Forces.

resultants are produced backwards to meet at L and are there resolved into their components LM, LN, etc. This step is justified, for a force can be represented by a line of suitable length drawn from any point in its line of action. The four forces at L now give a resultant LM + LM′ parallel to the lines of action of OA and O′A′, for the forces LN and LN′ nullify each other. By producing LM to cut OO′ in O″, the point in OO′ through which the line of application of the resultant passes is determined.

Moment of a Force.—Let AB, Fig. 3·3, represent a force F completely, and let OL be the perpendicular from any point O upon AB. Then F . OL is called the moment or torque of the force about O. This moment is represented graphically by twice the area of the Δ OAB, for F . OL is AB . OL which is twice the area of the triangle.

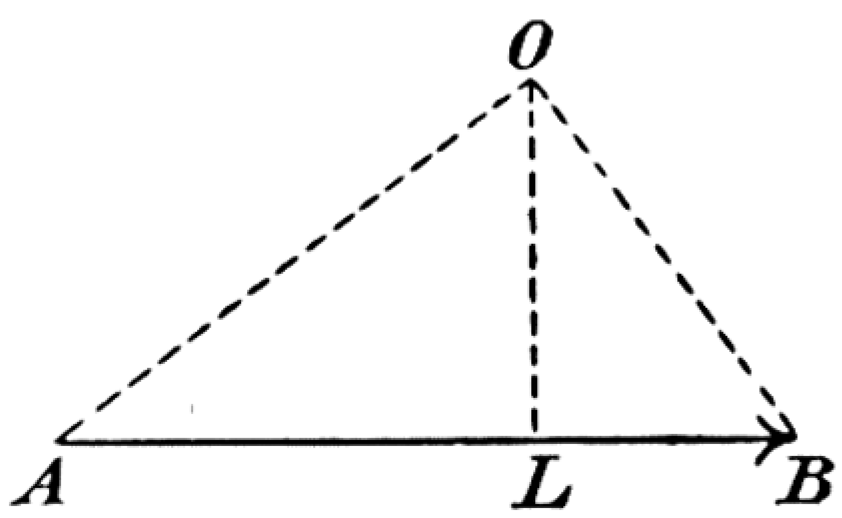


Fig. 3·3.—Moment of a Force.

Couples.—Two equal unlike parallel forces which are not collinear constitute a couple, the moment of which is equal to the magnitude of one of the forces multiplied by the perpendicular distance between the lines of action of the forces.

Centre of Gravity.—If a body is sufficiently ***small,*** its weight may be regarded as the resultant of the ***parallel*** forces acting on its constituent particles. It is found that for all such bodies there is some point, not necessarily ***in*** the body, but which has a definite position with regard to any point in the body taken as reference, and through which the line of action of the resultant of all these parallel forces passes irrespective of the actual position of the body. This point is called the ***centre of gravity*** [c.g.] of the body. The centre of gravity of any plane body, i.e. a lamina, such as a triangular sheet of metal of uniform thickness, may be found by suspending the body from any point and placing a plumb line immediately in front of the triangle and in such a position that it passes in front of the point of support. Under these conditions the plumb line indicates the line of action of the weight of the suspended body. This direction can then be marked on the triangle. The above procedure is repeated, the lamina being supported from another point. The centre of gravity is then that point at a distance one-half the thickness of the material behind the point of intersection of the lines indicating two positions of the plumb line.

The centre of gravity of a body such as a chair, or bird-cage, is more difficult to find since it is not easy to mark the position

of the plumb line which must invariably be used. It may be done, however, by attaching small pieces of plasticine to the cage and fixing straws therein so that the extremities of the straws touch the plumb line. The extremities of the straws are then joined by a silk thread attached by means of glue. A second determination gives the position of the centre of gravity, for it will be the point at which the two silk threads intersect.

Stable and Unstable Equilibrium.—Whenever a body is in statical equilibrium the resultant force upon it must be zero, but the nature of the equilibrium is not always the same. To illustrate these remarks let us consider the equilibrium of a sphere resting in turn on a concave, a convex, and a flat surface as shown in Fig. 3·4. When the sphere is given a slight displacement from its zero position on a concave surface it tends to return to this position as soon as the constraining force is removed. The equilibrium is said to be ***stable.*** The equilibrium, however, when the sphere rests on a convex surface is ***unstable,*** because if the sphere experiences even a very small displacement it never returns of its own accord to its former position. In the third case when the sphere rests on a flat surface the equilibrium is called ***neutral*** because the body may be at rest at any point on the surface.

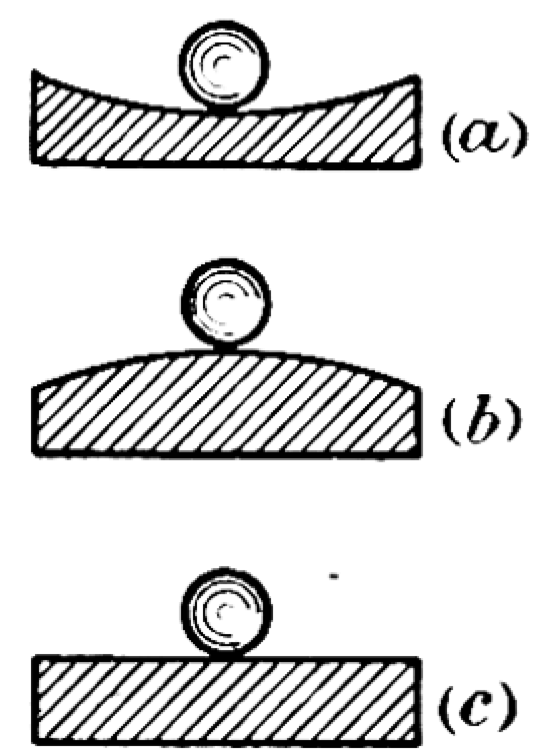


FIG. 3·4.—Types of Equilibrium.

Machines.—As a result of experience man has found that he can work better in some positions than in others; it is always more convenient to pull a rope downwards rather than upwards, and it is generally more convenient to apply a small force through a given distance when it would be impossible, for example, to apply a force ten times as great through one-tenth that distance. Hence it is desirable to have a contrivance for changing (*a*) the point of application, (*b*) the magnitude of a force and (*c*) its direction. ***A mechanical arrangement whereby a force acting at one point is made available at another point under different conditions as regards its magnitude and direction is known as a simple machine.*** If F is the effort or force necessary to be applied to a given machine to overcome the weight W of a load which it carries, $\frac{W}{F}$ is called the ***mechanical advantage*** of the machine.

Another quantity often evaluated in connexion with a machine is its ***velocity ratio.*** This is defined as the ratio of the distance through which the point of application of the effort moves to the

distance through which the point of application of the resistance or weight moves in the same time, i.e.

$$\text{Velocity ratio} = \frac{\text{distance through which F moves}}{\text{distance through which W moves}}.$$

If a machine is perfectly smooth and no energy is used in moving the component parts, the work done against W is equal to the work done by F,

i.e.
$$\text{W} \times \text{distance through which W moves}$$
$$= \text{F} \times \text{distance through which F moves}.$$

Hence,
$$\frac{\text{W}}{\text{F}} = \frac{\text{distance through which F moves}}{\text{distance through which W moves}},$$

or the mechanical advantage of a machine in which there is no friction and in which the parts have no weight is equal to its velocity ratio. In any actual machine the mechanical advantage must be determined experimentally but in simple machines it is possible to calculate the velocity ratio.

The efficiency of a machine is defined by the equation

$$\text{Efficiency} = \frac{\text{work done by the machine}}{\text{work done by the effort}}$$
$$= \frac{\text{load (W)} \times \text{distance load moves}}{\text{effort (F)} \times \text{distance effort moves}}$$
$$= \frac{\text{W}}{\text{F}} \div \text{velocity ratio, i.e.}$$
$$\text{Efficiency} = \frac{\text{mechanical advantage}}{\text{velocity ratio}}.$$

The Principle of Virtual Work.—Mechanical problems, especially those dealing with simple machines, i.e. machines without friction and in which no work is required to move the components, may be solved by a principle first pointed out by Stevinus in connexion with pulleys. He noticed that when a load of weight mg or W is raised by a cord passing over a single fixed pulley, that the effort is equal to the weight and that the point of application of the effort descends through a vertical distance equal to that through which the weight is raised. In the instance of a single movable pulley, the effort is only one-half of the weight of the load raised, but its point of application moves through twice the distance. Stevinus argued that this principle applied to all simple machines and wrote 'Ut spatium agentia ad statiam patentis, sic potentia patentis ad potentiam agentis,' a free translation of which is 'What is gained in power is lost in speed.' A better statement of this principle is that mechanical advantage is always gained at a proportionate diminution in speed.

In 1717 BERNOULLI, an eminent mathematician, extended the above principle to all cases of equilibrium. He maintained that if any number of forces acting on a body undergo infinitely small displacements consistent with the configuration of the system, then the total work done is zero, i.e.

$$\Sigma F \cos \alpha . \ \delta s = 0$$

where δs is the displacement of the point of application of F, and α is the angle between F and δs. The necessity for the displacements to be infinitely small follows at once from the fact that if they are finite the system may assume another configuration in which equilibrium is only maintained under conditions different from those for the given system. This principle, the so-called principle of virtual work, will be used in discussing some of the problems which follow.

Since no machine is without friction, etc., the principle of work, as here used, only allows us to calculate $\frac{W}{F}$ on the assumption that the machine is ideal, i.e. it gives us the velocity ratio in all cases but the mechanical advantage only if the machine is ideal.

Levers.—One of the simplest forms of machine is the lever, of which there are three classes according to the position of the

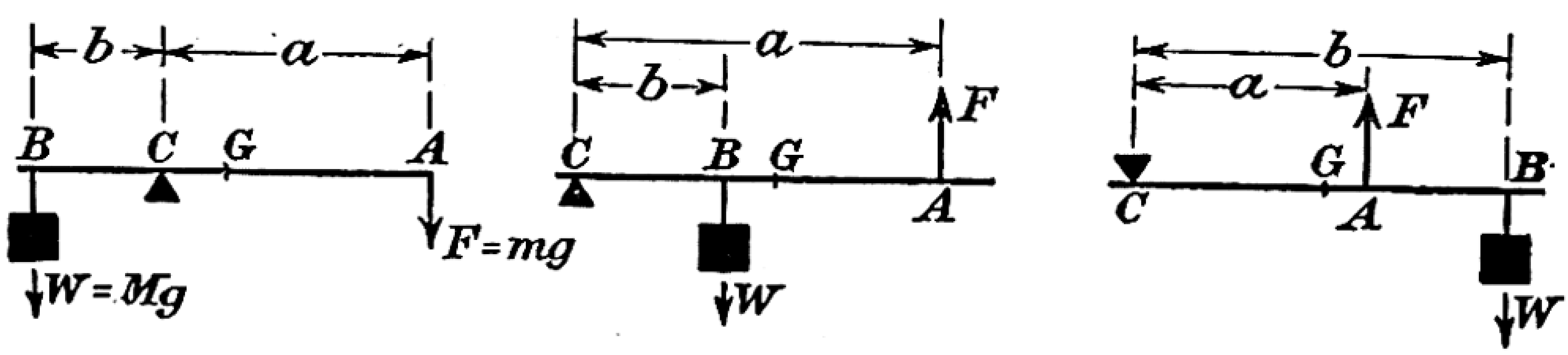


FIG. 3·5.—Levers.

point or ***fulcrum*** about which they turn. The three classes are shown in Fig. 3·5. In addition to the forces F and W there is the reaction at the fulcrum C, and since there is equilibrium the reaction must be equal to the algebraic sum of F and W. In all three instances when the levers are in equilibrium

$$F . AC = W . BC.,$$

or

$$\frac{W}{F} = \frac{a}{b}.$$

This is the mechanical advantage in so far as friction and the weight of the lever are negligible. The velocity ratio is calculated as follows :—

If the beam rotates through an angle θ, the point of appli-

cation of the effort moves a distance $a \sin \theta$ while that of the load moves a distance $b \sin \theta$, i.e. the velocity ratio is $\frac{a}{b}$.

In the above it has been assumed that friction and the weight of the lever are negligible. If the weight is W_1, and acts at G, the condition for equilibrium in the first class is

$$W \,.\, BC = W_1 \,.\, GC + F \,.\, AC.$$

Similar expressions are easily written down for the other two classes of lever.

The Balance.—The physical or analytical balance, the main features of which are shown in Fig. 3·6, consists of a light but rigid metal beam, supported so that it may rotate in a vertical plane

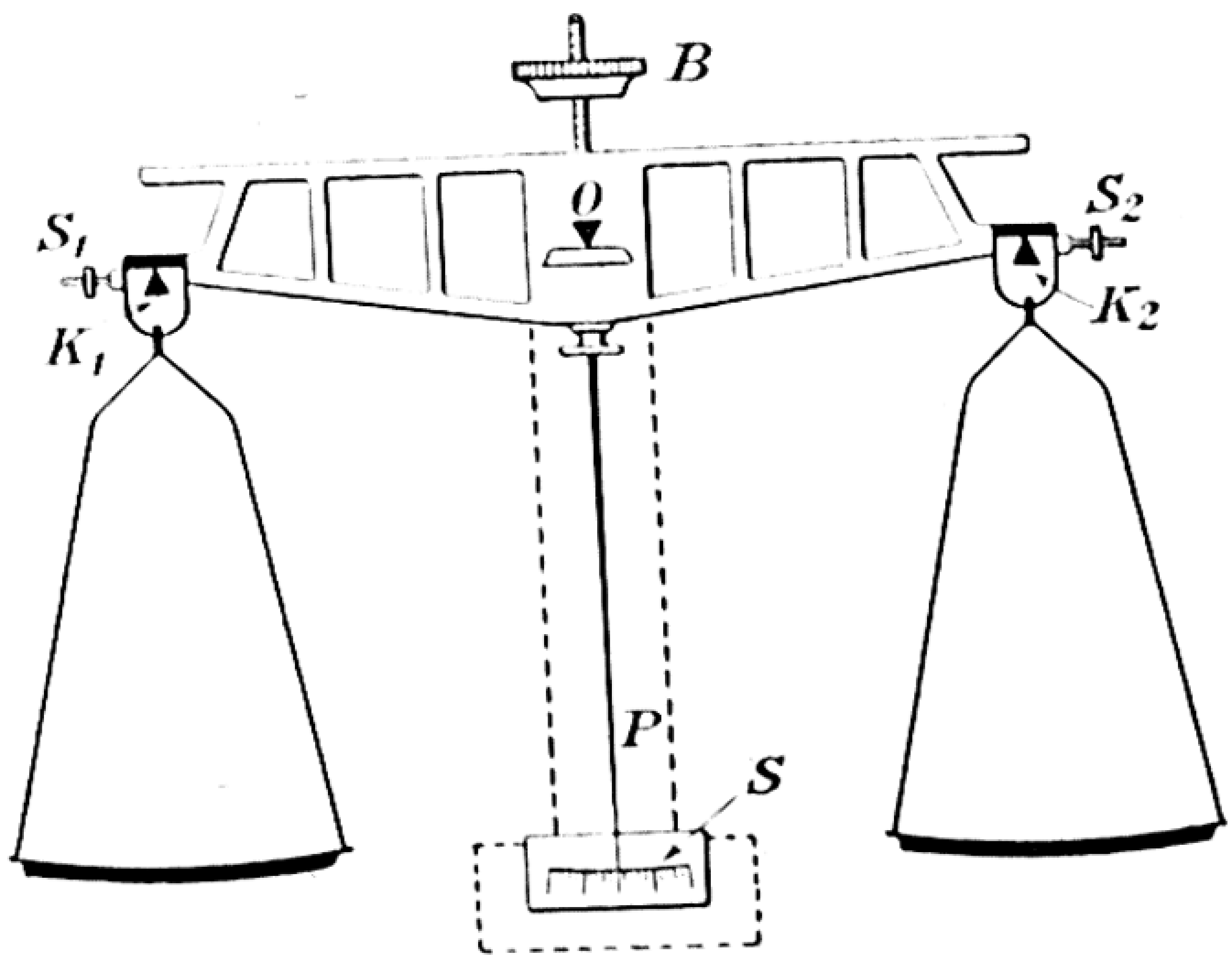


FIG. 3·6.—Main Features of a Physical Balance.

about a horizontal axis vertically above its centre of gravity. Pans are suspended from the extremities of the above beam and turn freely about axes parallel to its axis of rotation. The beam has three agate knife-edges: the central one, O, edge pointing downwards, supports the beam when it rests upon an agate plate attached to the pillar (indicated by the broken lines) of the balance, while the outer ones, edges pointing upwards, carry the agate plates to which the scale-pan supports are fixed. The whole is enclosed in a glass case to protect it from air currents and rapid temperature changes, the air within the case being kept dry by means of some desiccating agent such as concentrated sulphuric acid. When the balance is not in use the knife-edges are shifted slightly so that they are not in contact with the agate plates: in this way the knife-edges are kept in working order for longer periods. The position of the balance beam is defined by a long metal pointer, P, rigidly attached to it, its length being normal to the line joining the outer

knife-edges. The observation of the motion of the pointer, and consequently that of the beam, is facilitated by means of a scale S fixed to the pillar of the balance. Small masses, S_1 and S_2, capable of moving on screws attached to the beam, enable the balance to be adjusted. When the balance is in proper adjustment the pointer should swing through equal distances on either side of the central mark of the scale S when the line K_1K_2 is horizontal. The bob B, also moving on a screw attached to the beam, permits the position of the centre of gravity of the beam and its attachments to be raised or lowered with respect to the central knife-edge O.

Simple Theory of the Static Equilibrium of an Equi-arm Balance.—In principle the common balance is simply a lever of

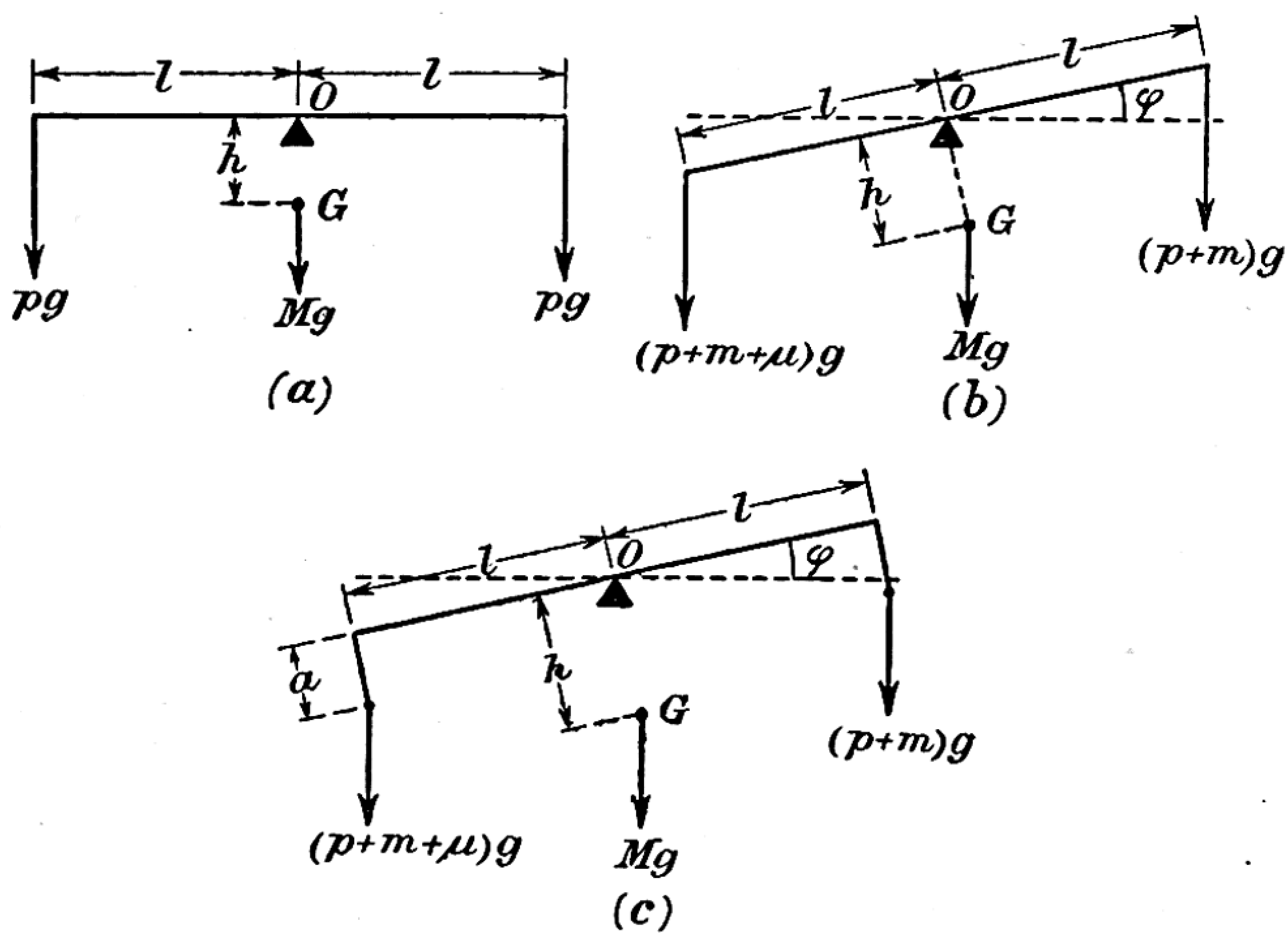


FIG. 3·7.—Sensitivity of a Balance.

the first class in which the two arms are equal and the central knife-edge O, Fig. 3·6, is the fulcrum. As a first approximation let us assume that the effective lengths of the arms are equal and invariable and that the three knife-edges are coplanar, horizontal and parallel to one another, as suggested in Fig. 3·7 (a). Let M be the mass of the beam, and its attachments, while G is their common centre of gravity. Further, let p be the mass of each pan and suppose that a mass m is placed in one pan, while $(m + \mu)$ is the mass in the other pan, where μ is a small mass. Let l be the length of each arm and h the distance OG. When the beam is loaded as above, let it take up an equilibrium position inclined

at a small angle ϕ to the horizontal—cf. Fig. 3·7 (*b*). Then, if g is the intensity of gravity, by taking moments of forces about O, we have

$$(p + m + \mu)gl \cos \phi = Mg\, h \sin \phi + (p + m)gl \cos \phi$$

$$\therefore \mu l \cos \phi = Mh \sin \phi$$

$$\therefore \tan \varphi = \frac{\mu l}{Mh}.$$

Since ϕ is small, $\tan \phi$ may be replaced by its circular measure ϕ, so that $\phi = \dfrac{\mu l}{Mh}$.

Now the ***sensitivity*** of a balance is defined as the change in ϕ caused by increasing μ by a given small amount, usually taken as 10^{-3} gm., i.e. if $\delta\phi$ is the change in ϕ thus caused,

$$\phi + \delta \phi = \frac{(\mu + 10^{-3})l}{Mh},$$

or

$$\delta\phi = \frac{10^{-3}l}{Mh}.$$

Thus, for the ideal balance, in which the knife-edges are coplanar, the sensitivity is independent of the load in the pans. From the above expression for the sensitivity it follows that, for a given balance, it may be varied by altering h, i.e. the position of G with respect to O. This, as already mentioned, is effected by means of the bob B, Fig. 3·6.

When the knife-edges are not coplanar let the outer edges, when the beam is at rest, be at a depth a below the horizontal plane through O—cf. Fig. 3·7 (*c*). Then, as before,

$$(p + m + \mu)g(l \cos \phi - a \sin \phi) = Mg\, h \sin \phi + (p + m)g(a \sin \phi + l \cos \phi),$$

i.e. $$\cos \phi[(p + m + \mu)l - (p + m)l] = \sin \phi[Mh + a(p + m) + a(p + m + \mu)].$$

Since $\phi \to 0$, $\cos \phi \to 1$ and $\sin \phi \to \phi$.

$$\therefore \phi \simeq \frac{\mu l}{Mh + a[\mu + 2(p + m)]}$$

$$\simeq \frac{\mu l}{Mh + 2a(p + m)} \quad [\because \frac{\mu}{p} \to 0].$$

The sensitivity is therefore given by

$$\delta\phi = \frac{10^{-3}l}{Mh + 2a(p + m)}.$$

Thus, if the three knife-edges are not coplanar, the sensitivity decreases with increasing load: since all beams are deformed

slightly by the load, so that the knife-edges never remain coplanar when the balance is loaded [in fact a varies with the load, also], the sensitivity of all balances decreases with the load.

If a balance is to be classified as a good one, it must possess the following characteristics :—

(*a*) Its indications must be reliable; i.e. the beam must be horizontal when equal masses are placed in the two pans. This is secured by making the arms exactly equal in length and mass; the suspended pans must also be of equal mass.

(*b*) The balance must be sensitive, i.e. a small difference between the two masses compared must cause an appreciable deviation of the beam from its zero position, i.e. ϕ must be relatively large. This is obtained by making M and h small. Hence the beam must be long and light, and have its centre of gravity near to O.

(*c*) A good balance must be stable, i.e. it must not suffer any change in shape, e.g. by bending of the beam, etc. For this reason the sensitivity cannot be increased indefinitely, for such a condition can only be attained by using a light beam, whereas the beam must be fairly massive if it is to be rigid. Evidently these conditions are at variance and, in practice, a compromise must be effected.

(*d*) The period of swing should be short, so that 'weighings' may be made rapidly—unfortunately this implies a less sensitive balance, so that again a compromise is made.

(*e*) The balance must be stable, i.e. when the balance is in equilibrium it must return to its zero position after being deflected.

(*f*) In addition the knife-edges [which, it must be noted, are always fixed to the beam] should be parallel and lie in the same horizontal plane. This latter condition is essential if the sensitivity of the balance is not to vary with the load in the pans.

A Micro-Balance.—Small masses, such as drops of liquid absorbed in bits of filter paper, or small quantities of powder, may be estimated by means of a micro-balance shown in Fig. 3·8. A light thread, H, has its extremities attached to the lower ends of a pair of nearly vertical rods 2 ft. apart, each of which is pivoted at a point just above its centre of gravity. Thus, a very light load suspended at the middle

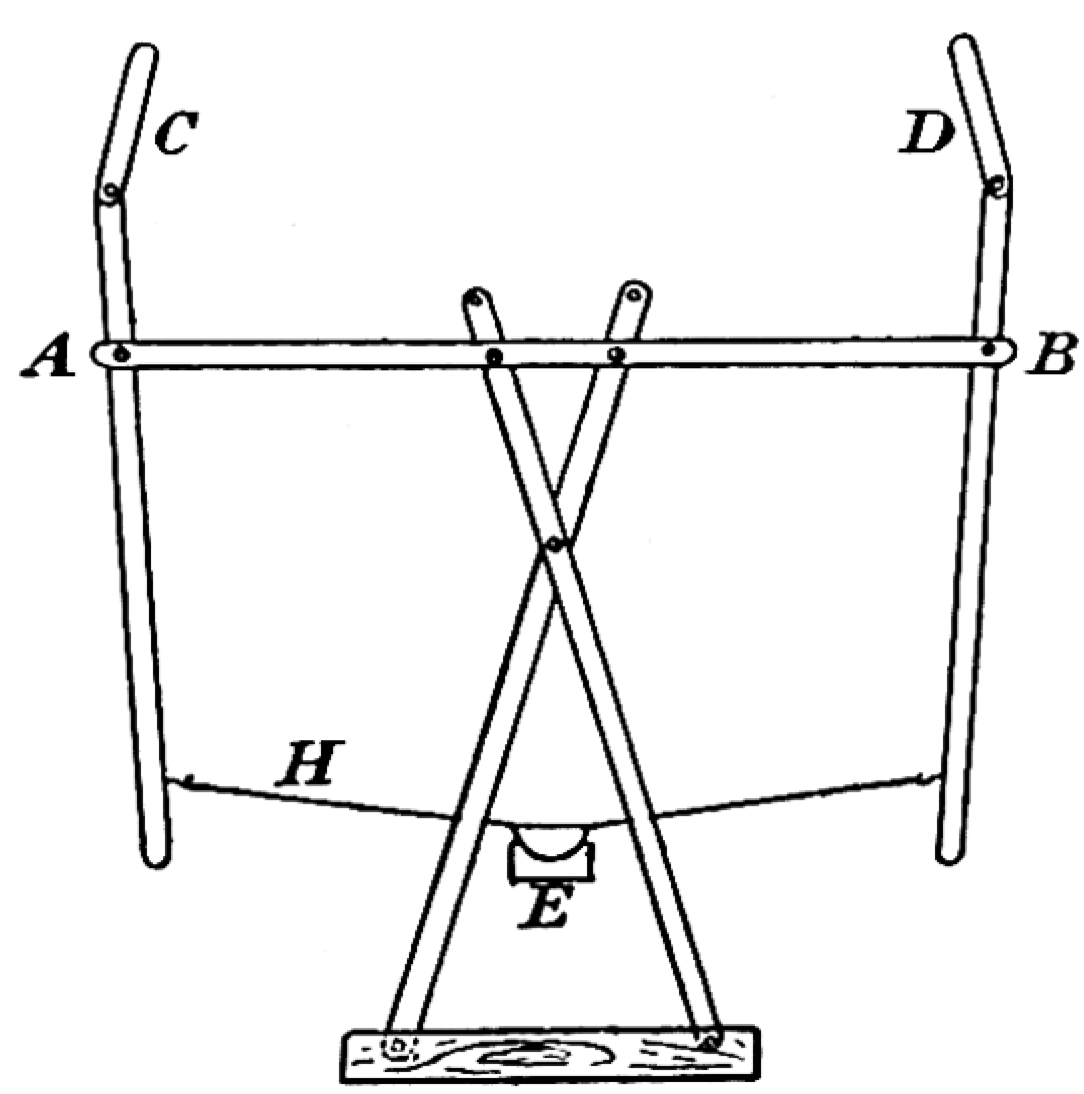


FIG. 3·8.—A Micro-Balance.

of the thread causes a considerable depression of that point. The apparatus is calibrated by observing the depression for a known load. The contrivance can be constructed out of 'Meccano' parts. The friction at the fulcrums A and B is reduced by using short glass tubes as the supports for the uprights. By making the uprights in two parts as shown and moving the upper portions C and D the sensitivity may be altered considerably. The 'pan' of the balance consists of a small circular disc bent across one of its diameters so as to form a clip which can be suspended from the thread, as at E. Small objects can then be supported between the jaws of this clip. Such an instrument as this has many uses, especially in the study of bacteriology. It was originally designed for use in Flanders, during the war of 1914–1918.

The Single Movable Pulley.—In this very simple type of machine a string, fastened at one end to a beam, passes round a pulley, K, Fig. 3·9, carrying a load of weight W. The portions of the string passing round the movable pulley are parallel to one another. The effort, or force, F, necessary to raise the load, is applied at the free end of the string which, for convenience, may pass round a fixed pulley, L. To determine the force required to maintain W in equilibrium a spring balance, S, is placed as indicated in the diagram. From observations made with such an apparatus it is soon realized that, if friction and the weight of the pulley be neglected, the tension in the string, which is measured by S, is one half the weight of W; this means that one half the

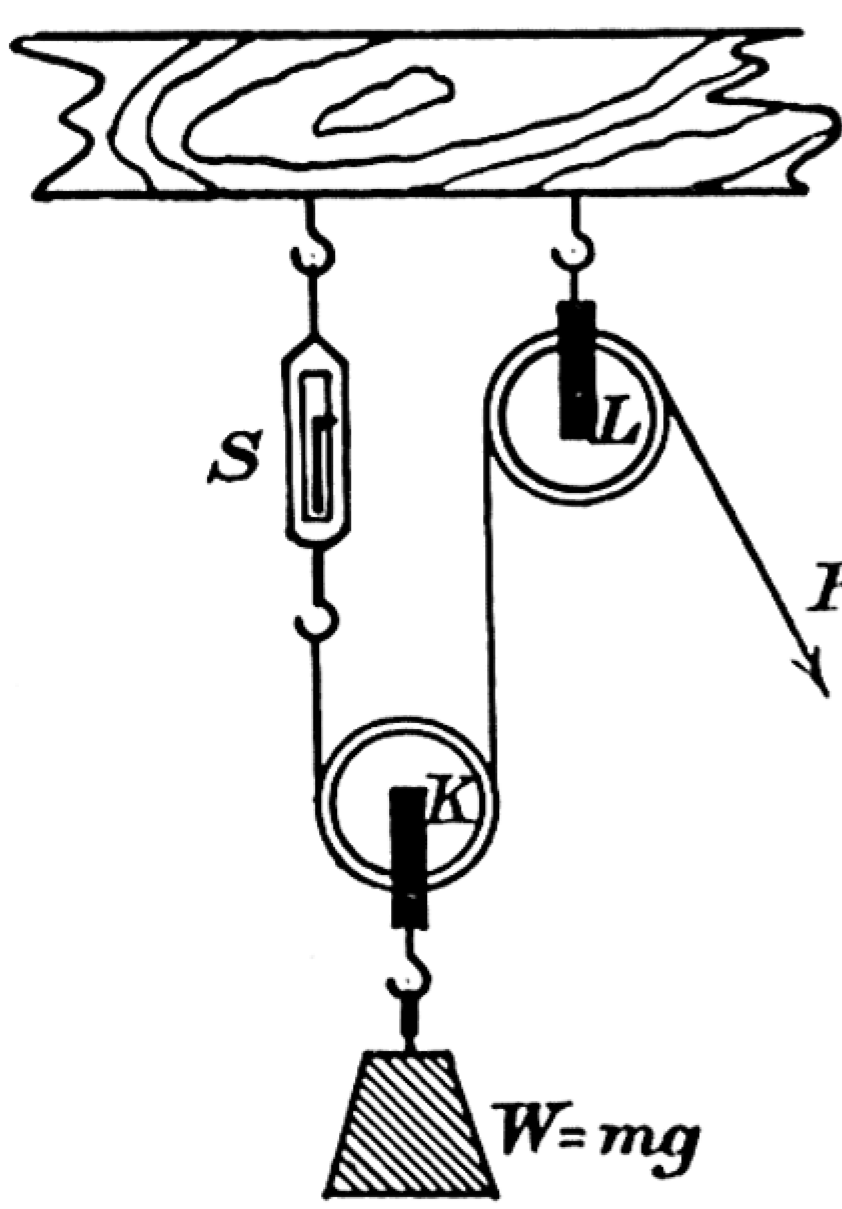


Fig. 3·9.—A Movable Pulley.

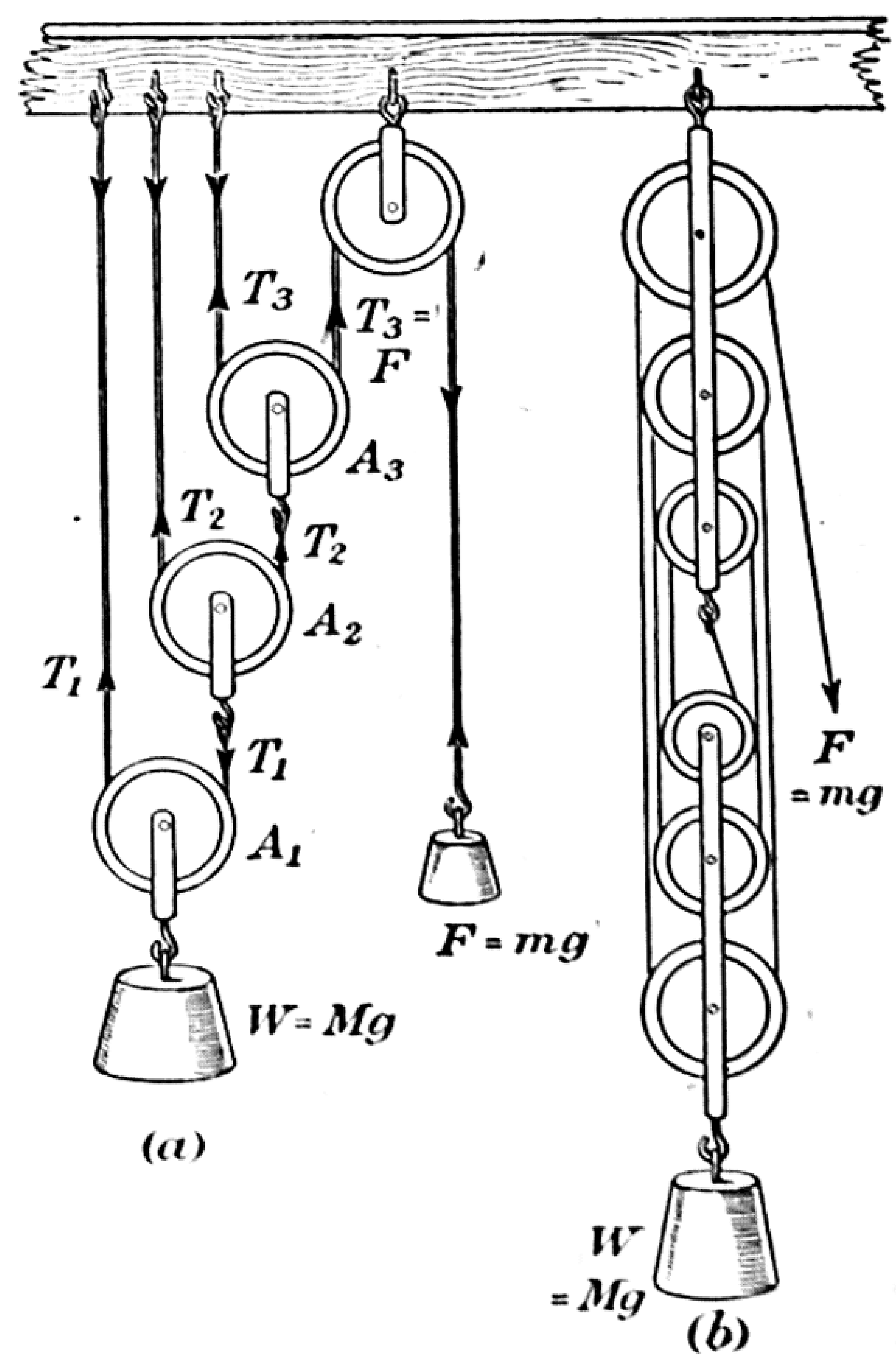


Fig. 3·10.—Systems of Pulleys.

load is supported by the string attached to the beam and the second half by the string passing round the fixed pulley—the free end of this string may be held in any convenient direction. The fact that a movable pulley-wheel with parallel strings reduces by one half the effort required to raise an object is a principle which may always be applied to such pulleys when they are free to move.

Of the various systems of pulleys with parallel strings, and the ways in which pulleys may be combined to form a machine, only two will be considered; they are shown in Fig. 3·10 (*a*) and (*b*).

In the ***Archimedean or First System of Pulleys,*** Fig. 3·10 (*a*), a separate string passes round each pulley. If the load W ascends a distance x, the string round A_1 is shortened by an amount x on each side, so that A_2 moves a distance $2x$. Similarly A_3 moves a distance $2 \times 2x = 2^2x$ and the point at which F is applied descends a distance 2^3x. The velocity ratio is therefore $2^3 = 8$. In the case of n movable pulleys the velocity ratio is 2^n.

To calculate the mechanical advantage of the system in the absence of friction and neglecting the weights of the pulleys we make use of the fact that the work done on W is equal to the work done by F, i.e. if there are three movable pulleys

$$W \cdot x = F \cdot 2^3x, \text{ or } \frac{W}{F} = 2^3.$$

Alternatively, let the tensions in the different strings be as shown. Then

$$T_1 = \tfrac{1}{2}W,\ T_2 = \tfrac{1}{2}T_1,\ T_3 = \tfrac{1}{2}T_2 = \tfrac{1}{8}W.$$

Since $T_3 = F$, $\dfrac{W}{F} = 8$, as above.

In the case of n movable pulleys the mechanical advantage, that is $\dfrac{W}{F}$, is 2^n.

In the above argument the weight of the pulleys has been neglected. Suppose that there are three movable pulleys, each of weight w. Then $T_1 = \tfrac{1}{2}(W + w)$;

$$T_2 = \tfrac{1}{2}[\tfrac{1}{2}(W + w) + w] = \tfrac{1}{4}W + \tfrac{3}{8}w;\quad T_3 = \tfrac{1}{2}[\tfrac{1}{4}W + \tfrac{3}{8}w + w]$$

$$\therefore F = T_3 = \tfrac{1}{8}W + \tfrac{11}{16}w.$$

$$\therefore \frac{W}{F} = 8 - \frac{11}{2}w\left(\frac{1}{F}\right).$$

Calling $\frac{W}{F}$, the mechanical advantage, y, and $\frac{1}{F} = x$, we have

$$y = 8 - \frac{11}{2}wx.$$

This equation suggests that if corresponding values of x and y are plotted, the resulting graph will be a straight line whose slope is $-\frac{11}{2}w$. Hence w can be determined.

In the ***Second or Common System of Pulleys*** there is only one continuous string and this passes round all the pulleys, its one end being fixed to the upper support, and the pull F applied at the other extremity [cf. Fig. 3·10 (b)]. If the load rises a distance x, the amount of rope 'set free' is $6x$, since each string supporting the lower block is shortened by an amount x. To keep the string taut the point of application of the effort must descend a distance $6x$: the velocity ratio is therefore 6. When there are n strings supporting the lower sheave pulley block the velocity ratio is n.

In actual practice the pulleys in each block in the common system are all concentric so that the two blocks can be drawn nearer together. The great disadvantage of these systems is that a long length of rope is required; this is avoided in the differential pulley.

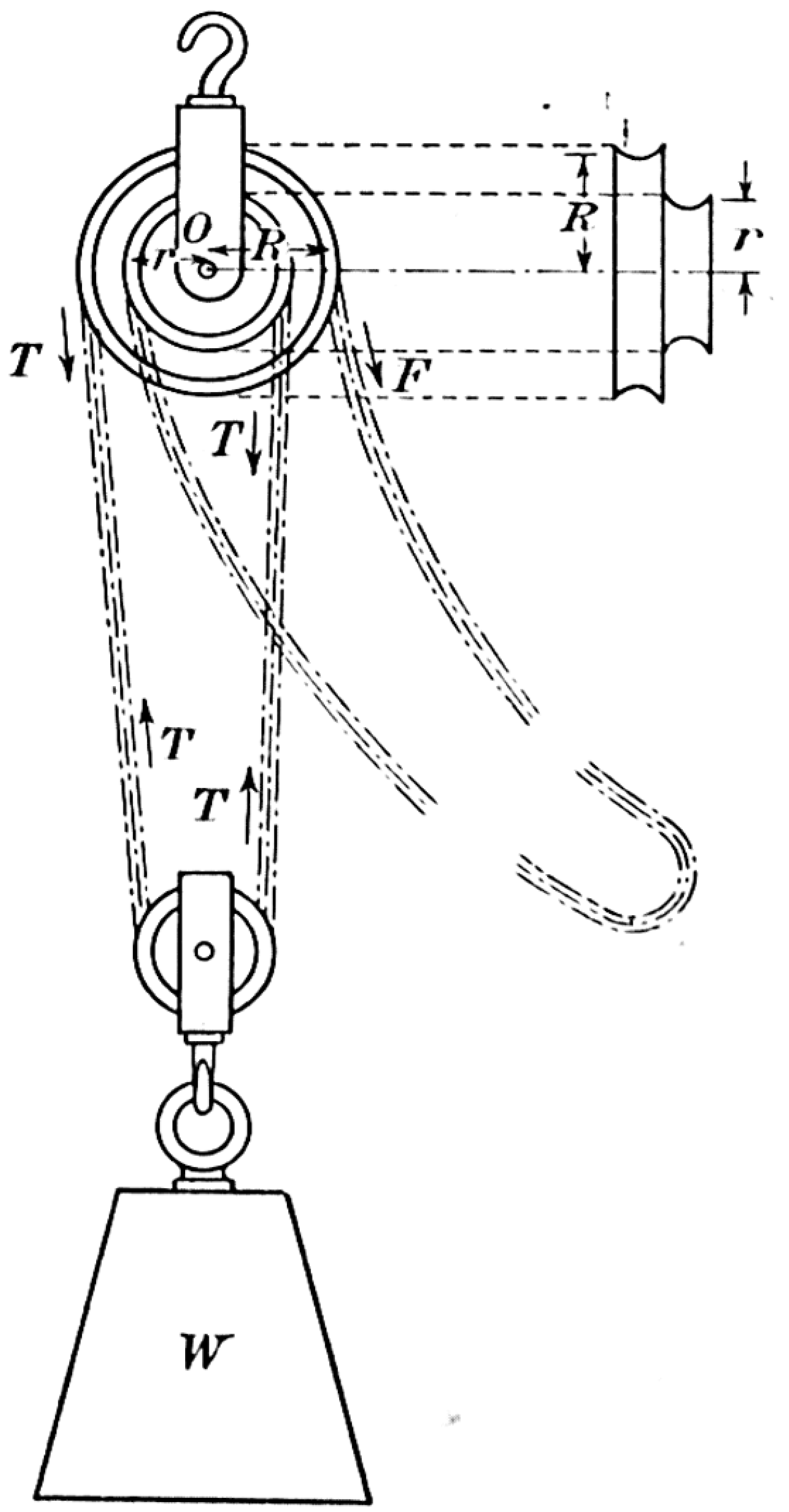


Fig. 3·11.—Weston's Differential Pulley.

Weston's Differential Pulley.—In this system the rope is replaced by an endless chain, slip being prevented by depressions in the grooves of the pulleys, and into these depressions fit the links of the chain. The system is represented in Fig. 3·11, in which the two pulleys of the upper block move

as one round a common axis. The effort F is applied as shown. If W is the load and T the tension in the string, the necessary condition for the equilibrium of the load, on the assumption that the machine is an ideal one, is $W = 2T$, whilst, by taking moments of forces round O, the relation

$$F . R + T . r = T . R$$

is obtained. Hence

$$F = T . \frac{R - r}{R} = \frac{1}{2}W . \frac{R - r}{R}.$$

The mechanical advantage, $\frac{W}{F}$, is therefore $\frac{2R}{R - r}$, and since the machine has an efficiency of 100 per cent, this fraction is also the velocity ratio.

The value for this ratio can be found for a differential pulley, even when it is not an ideal system, as follows :—

Suppose that the upper pulley block makes one complete revolution and that W rises. The length of chain wound in $= 2\pi R$, while that let out $= 2\pi r$. Therefore the length of chain actually supporting the lower pulley and W is shortened by an amount $2\pi(R - r)$, i.e. W rises a distance $\pi(R - r)$. Since the point of application of F descends a distance $2\pi R$, the velocity ratio is $\frac{2R}{R - r}$. Hence in this non-ideal machine the efficiency is

$$\frac{W(R - r)}{2FR}.$$

The Inclined Plane.—When a body S, Fig. 3·12, rests on a smooth inclined plane it is acted upon by two forces, the weight of the body acting vertically downwards, and the reaction of the plane on the body which is normal to the surface. The body will therefore move under the influence of the resultant of these forces unless it is constrained by some other force. The two cases which we shall study are when this third force, F, is either parallel to the line of greatest slope in the plane, or to the base of the plane—cf. Fig. 3·12 (*a*) and (*b*). The velocity ratio is $\frac{AB}{BC}$, i.e. cosec θ in the first instance and $\frac{AC}{BC}$, i.e. cot θ in the second. These expressions are valid independently of whether friction is present or not. For any actual plane, as indeed for any actual machine, the mechanical advantage $\frac{W}{F}$ must be determined experimentally. If the plane is smooth the mechanical advantage may be calculated as follows. To determine the magnitude of the effort F required to hold a

body of weight W on a smooth plane whose inclination is θ and when F is parallel to the line of greatest slope AB in the plane we resolve the forces acting on S along AB. This gives

$$mg \cos\left(\frac{\pi}{2} - \theta\right) = \text{F},$$

i.e.

$$mg \sin\theta = \text{F}.$$

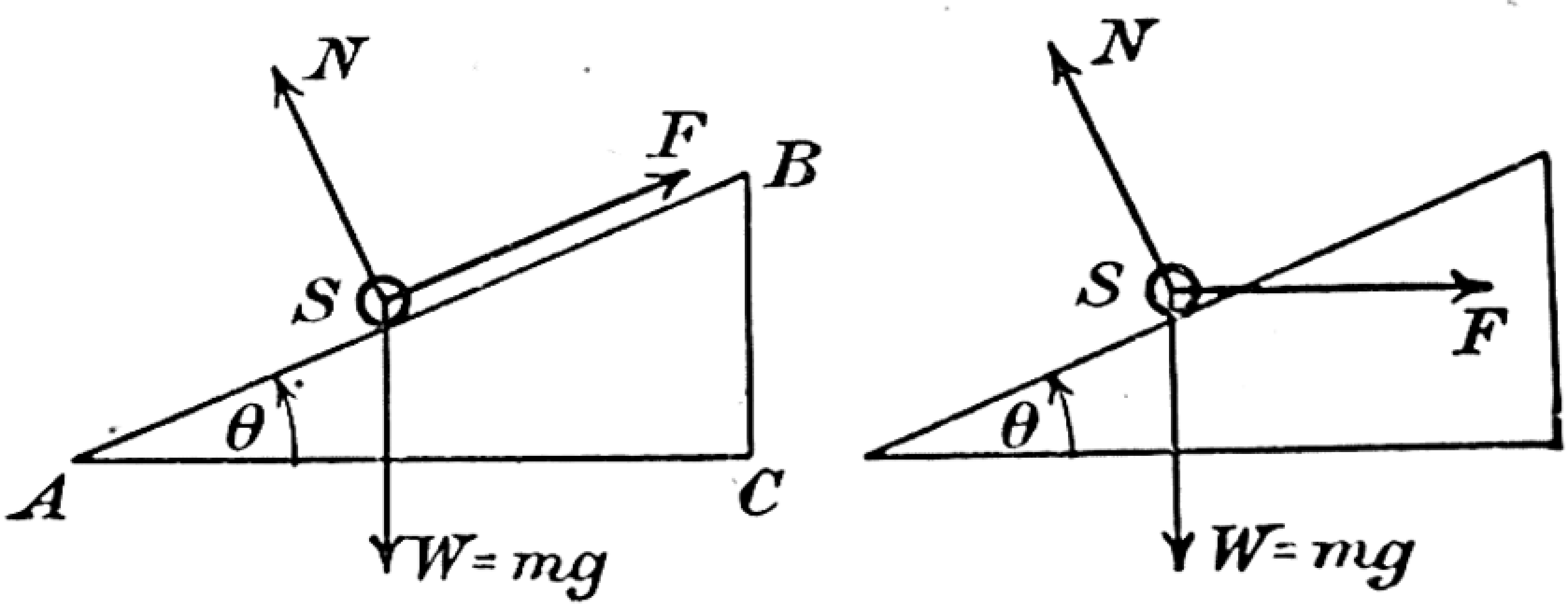


FIG. 3·12.—The Inclined Plane.

Similarly, by resolving forces perpendicular to the plane, we get the normal reaction, N, of the plane on the body, viz.,

$$\text{N} = mg \cos\theta.$$

In these equations it must be remembered that if m is expressed in pounds, F and N are in poundals. The more usual practice is to express the weight mg as W lb.-wt., when the above equations become

$$\text{W} \sin\theta = \text{F etc.},$$

where F and N are now measured in lb.-wt. Since $\frac{\text{W}}{\text{F}}$ is the mechanical advantage of the system it follows that this is equal to cosec θ in this instance; in the second it can be shown to be cot θ.

The Screw.—If a triangle PQR, Fig. 3·13, in which $\widehat{\text{QPR}}$ is equal

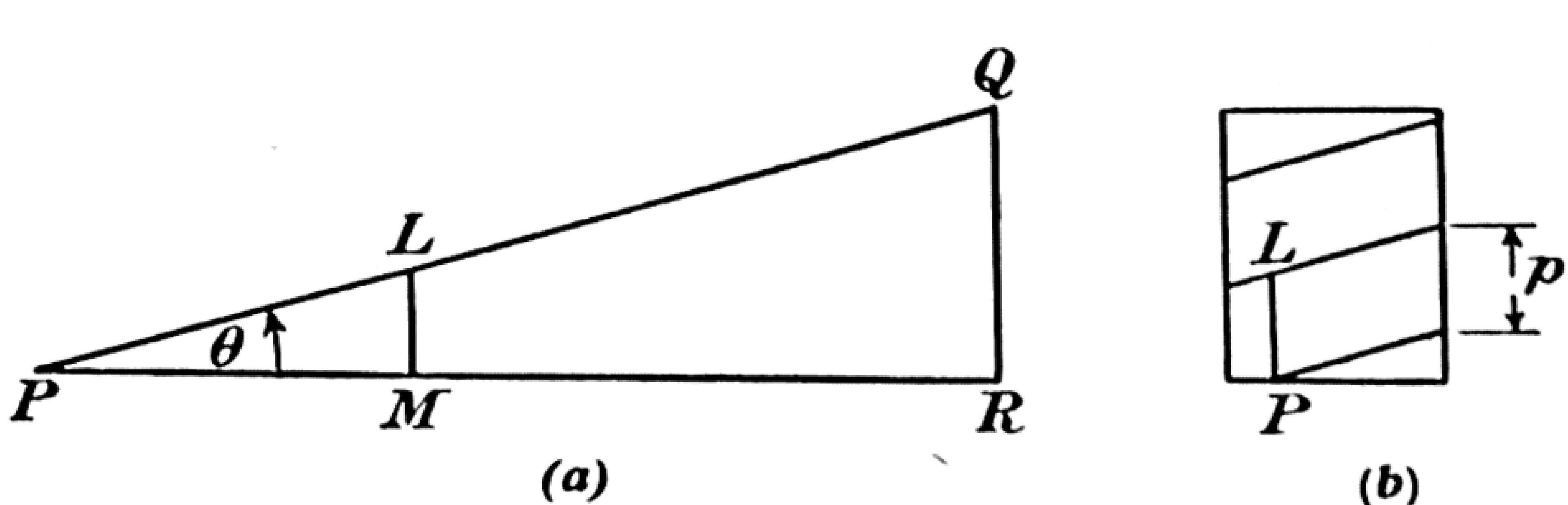


FIG. 3·13.—The Principle of a Screw.

to θ, is constructed out of thin paper or aluminium foil and wrapped round a right circular cylinder so that the base PR remains in a plane perpendicular to the axis of the cylinder the trace of the hypotenuse on the surface of the cylinder is a spiral. Let LM be a line perpendicular to the base of the Δ PQR such that when the paper

is round the cylinder the point M coincides with P, and L is vertically above P. Then, regarding the trace of the edge PQ as the thread of the screw, LM is the pitch of the screw. The $\widehat{QPR}$ is called the angle of the screw and it is clear from the diagram that $\tan\theta = \frac{LM}{PM} =$ pitch of screw $\div$ circumference of cylinder.

Actual screws differ from this ideal screw in that they always have a protuberant thread of metal or wood, etc. This enables the screw to work in a nut, but of course introduces so much friction that the mechanical advantage of a screw never approaches equality with its velocity ratio which we now proceed to obtain.

The Velocity Ratio and Efficiency of a Screw.—Let us suppose that we have a screw working in a nut and that the screw is supporting a load of weight W, as in Fig. 3·14, while a force F, which we assume to be in a horizontal plane, is applied to the end of the arm AB. Now when the arm AB has made one complete revolution the point of application of F has moved through a distance $2\pi r$ where r is the distance of B from the axis of the screw. Under the same circumstances the load W will have moved through a distance p, where p is the pitch of the screw. The velocity ratio is therefore $\frac{2\pi r}{p}$.

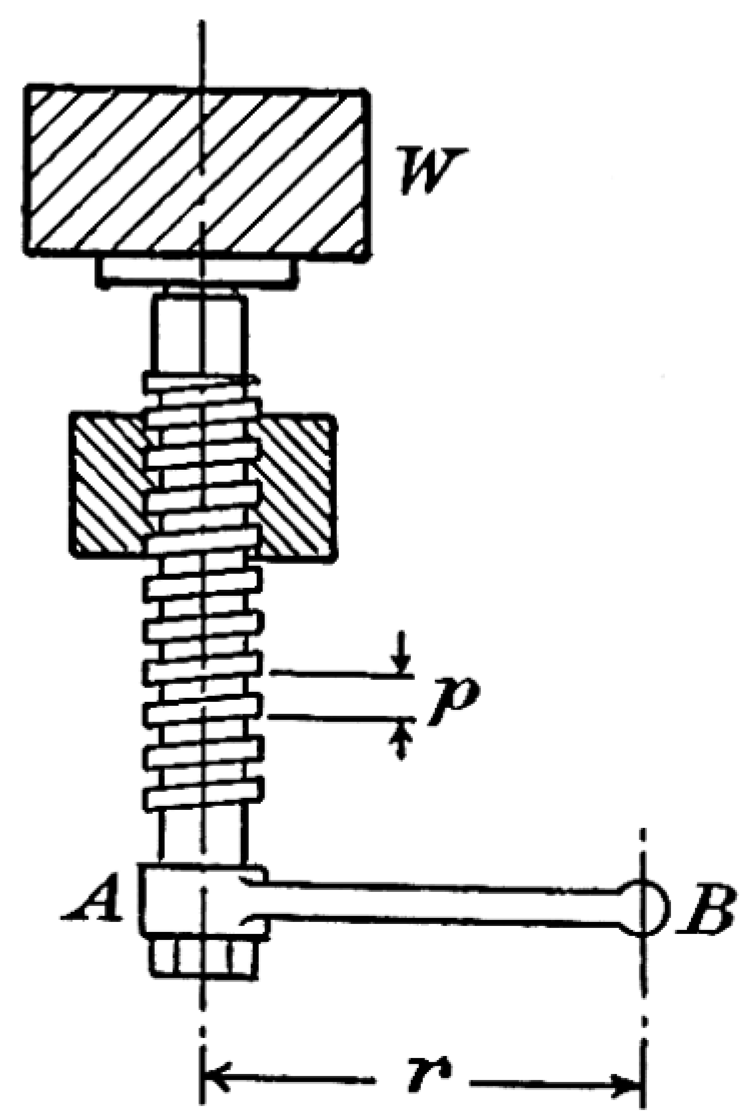


FIG. 3·14.—Mechanical Advantage of a Screw.

If the mechanical advantage, $\frac{W}{F}$, is determined experimentally, the efficiency is $\frac{W}{F} \cdot \frac{p}{2\pi r}$.

Weighing Machines.—The mass of a heavy load may be ascertained with the aid of a weighing machine the principle of which is indicated in Fig. 3·15. It consists of three levers ACD, EK, and LR respectively. The platform upon which the load is placed is attached to the lever EK, whilst the end D of the first lever carries a scale-pan. The fulcrum for the lever EK is not fixed but is attached to the lever LR moving about a fixed fulcrum R. If a load of mass M, and therefore weight Mg, is placed on the platform we may regard its weight as being distributed at the points E and K. The actual distribution will depend upon the position of the load on the platform, but let us suppose that there is a load mg at K so that the load at E is $(M - m)g$. The load mg at K can be replaced by a load

mg/n at L if LR $= n$. KR. Now the load at L may be considered to be acting at A, and may therefore be replaced by a load n times as large at B if AC $= n$. BC, i.e. the equivalent load at B would be mg. But the load at E may be replaced by an equal one at B so that

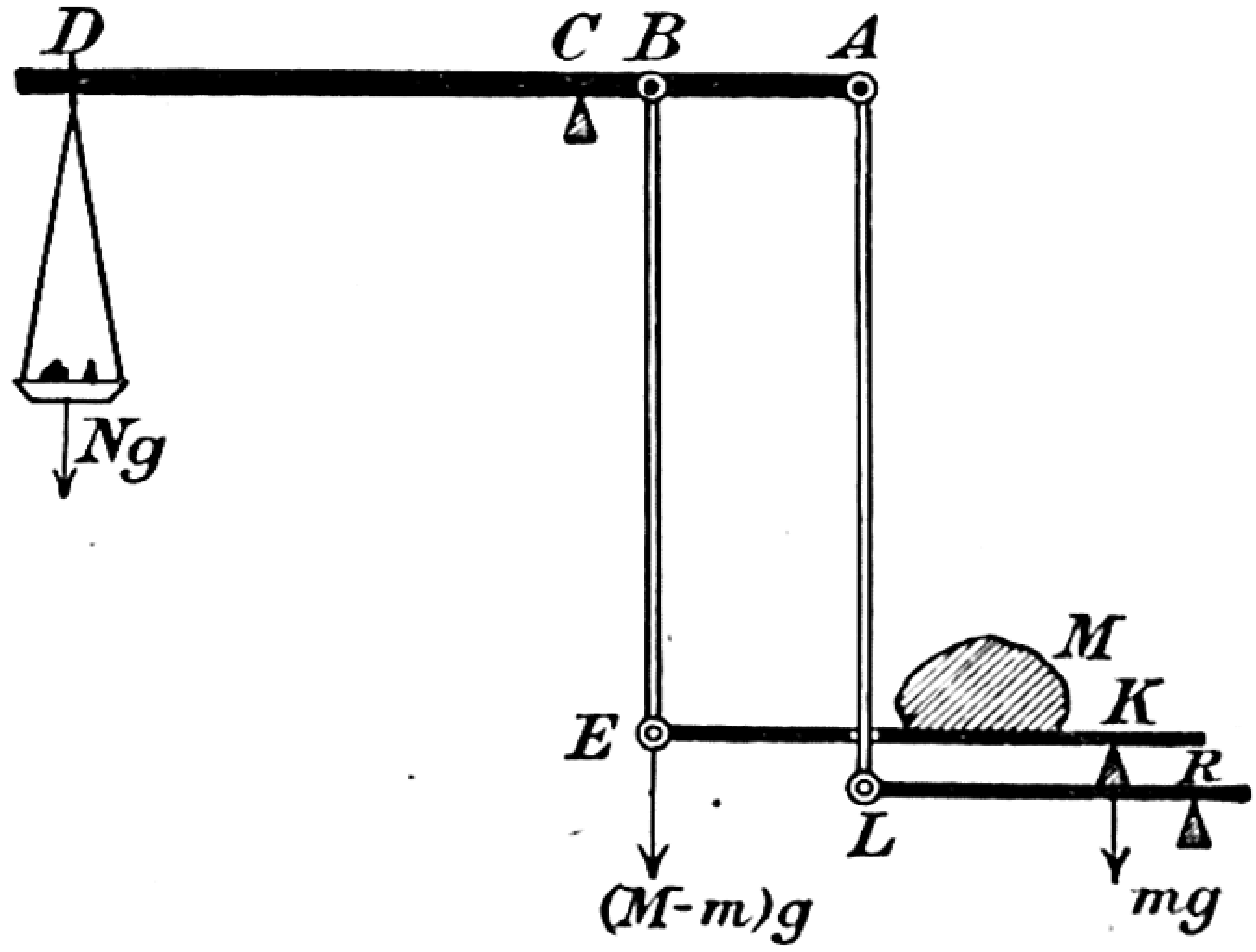


FIG. 3·15.—A Weighing Machine.

the total load at B is now Mg, and this is independent of the actual position of the object on the platform. To measure this load at A the length of the lever CD may be made 10 or 100 times that of AC. When this is done the mass N of the load on the scale-pan is the corresponding fraction of the mass M.

The Common or Roman Steelyard.—This is another machine for determining the mass of a heavy load, and consists of a long non-

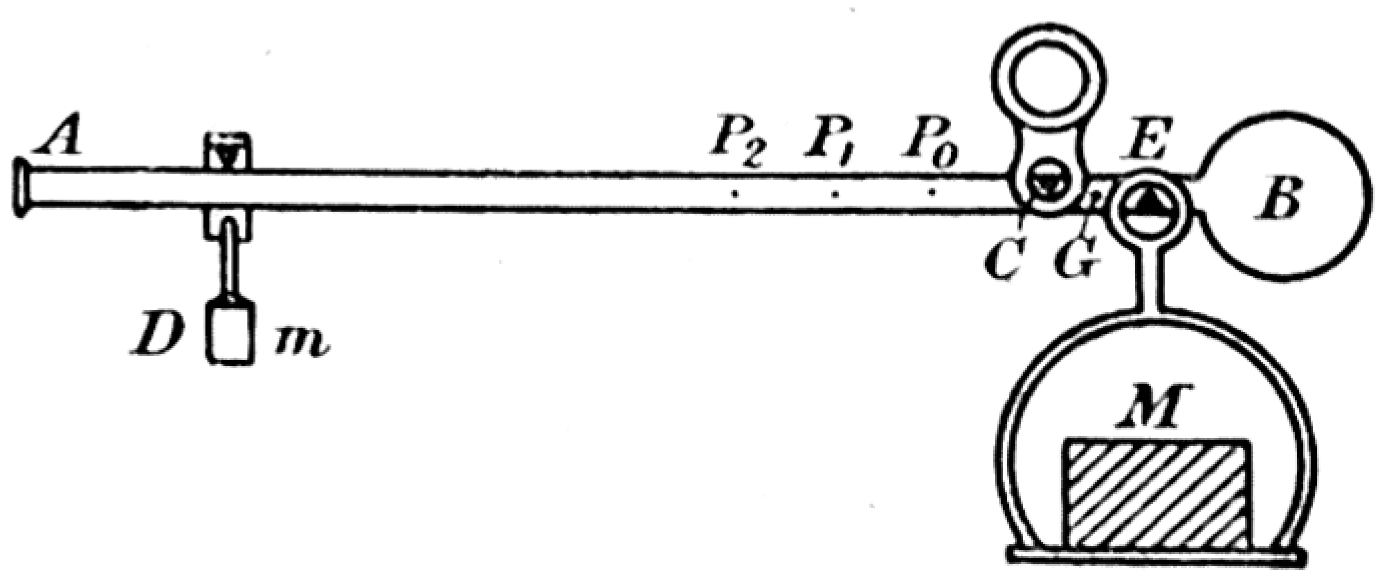


FIG. 3·16.—Common Steelyard.

uniform rod, AB, Fig. 3·16, movable about a fixed fulcrum, C, situated a little to the left of G, the centre of gravity of the bar. A hook, or scale-pan, hung from E, is used to carry the load of mass M, whilst D is a bob of mass m movable along AC. The point at which D must be placed to maintain the steelyard in a horizontal position enables one to determine the mass of the load M.

To calibrate the steelyard let P_0 be the position of D when the load is zero; this point is given by the equation

$$m \,.\, P_0C = \mu \,.\, CG,$$

where μ is the mass of the steelyard. [All masses are expressed in stones, where 1 stone = 14 lb.] When the load in the scale-pan is 1 stone let D be at P_1. The position of P_1 is determined by

$$m \,.\, P_1C = \mu \,.\, CG + (1 \times EC).$$

Subtracting the first equation from this we have

$$m \,.\, P_0P_1 = 1 \times EC.$$

Similarly, when the load is 2 stones the position of D is given by

$$m \,.\, P_0P_2 = 2 \times EC.$$

We see therefore that this instrument may be graduated by engraving marks upon the bar such that their common distance apart is equal to EC/m, the zero division being at P_0 as defined above.

The Danish Steelyard.—This consists of a bar, AB, Fig. 3·17, terminating in a sphere at B. The other extremity of the bar carries a scale-pan to receive the load whose mass is required. The pan is fixed, so that the mass of the load is determined by observing the point in the rod about which it balances. To graduate the steelyard let m be the mass of the whole including the pan, and let G be the centre of gravity. If C is the fulcrum when the load in the pan has a mass M, by taking moments of forces about C we have

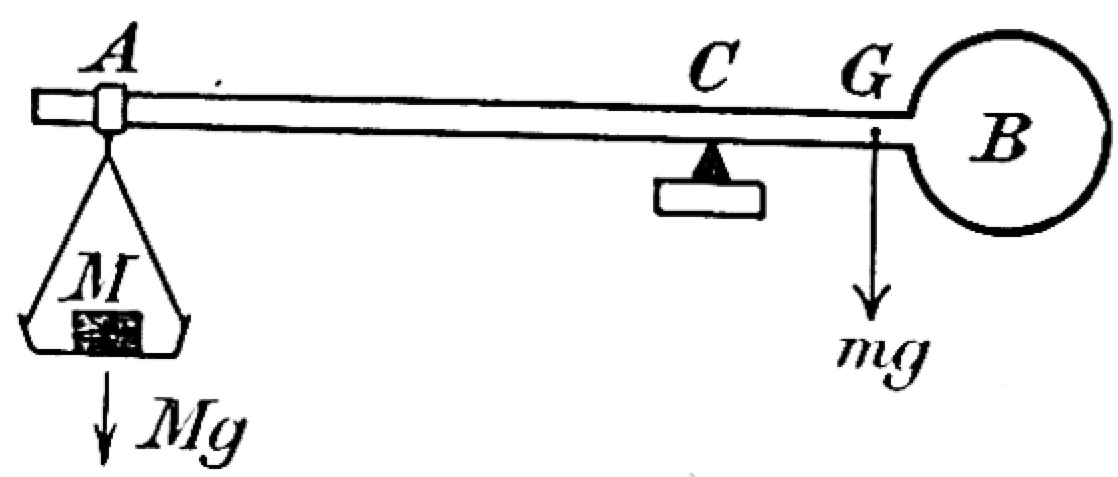


FIG. 3·17.—Danish Steelyard.

$$Mg \,.\, AC = mg \,.\, GC = mg \,.\, (AG - AC).$$

Therefore $$AC = \frac{m}{M + m} \,.\, AG.$$

This equation indicates that to graduate the steelyard it should first be balanced about its centre of gravity, i.e. G is found. Let us further assume that the mass m is 1 stone. The middle point of AG is the fulcrum when the load in the pan is 1 stone. Similarly, when the load is increased to 2 stones the fulcrum must be at a distance $\frac{1}{3}$ AG if the whole is in equilibrium. We therefore see that if the load is n stones the point of balance must be such that

$$AC = \frac{1}{n+1} AG.$$

Friction.—Hitherto it has been supposed that the surfaces of bodies in contact have been perfectly smooth, so that the reaction of one on the other was always directed along the common normal to the surfaces at the point of contact. In practice this condition is only satisfied if there is no tendency for relative motion between the surfaces: when there is such a tendency, forces are called into play and oppose the motion. These forces are due to ***friction*** between the surfaces in contact.

The Laws of Static Friction.—The effects of friction were investigated experimentally by Coulomb in the following manner. A, Fig. 3·18 (*a*), is a board resting on a horizontal table. B is a slider which could be suitably weighted in order to vary the

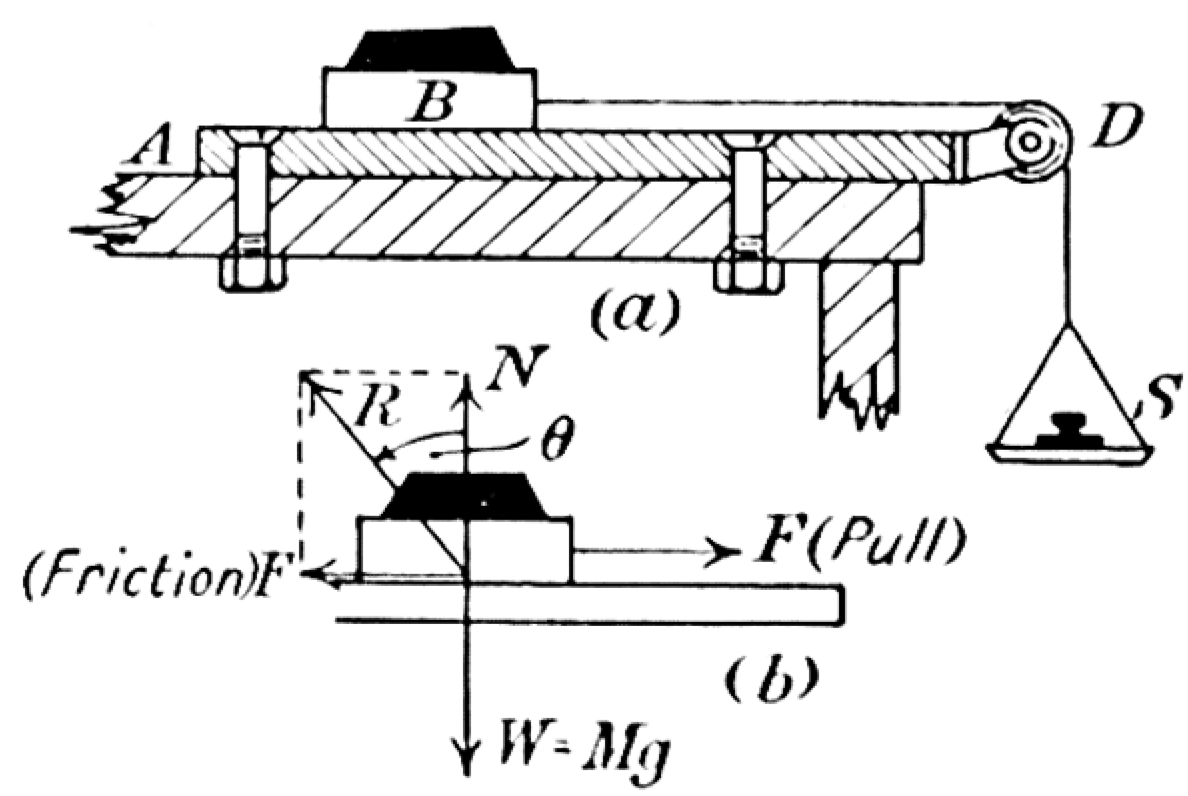


Fig. 3·18.

Coulomb's Apparatus for Investigating the Laws of Static Friction.

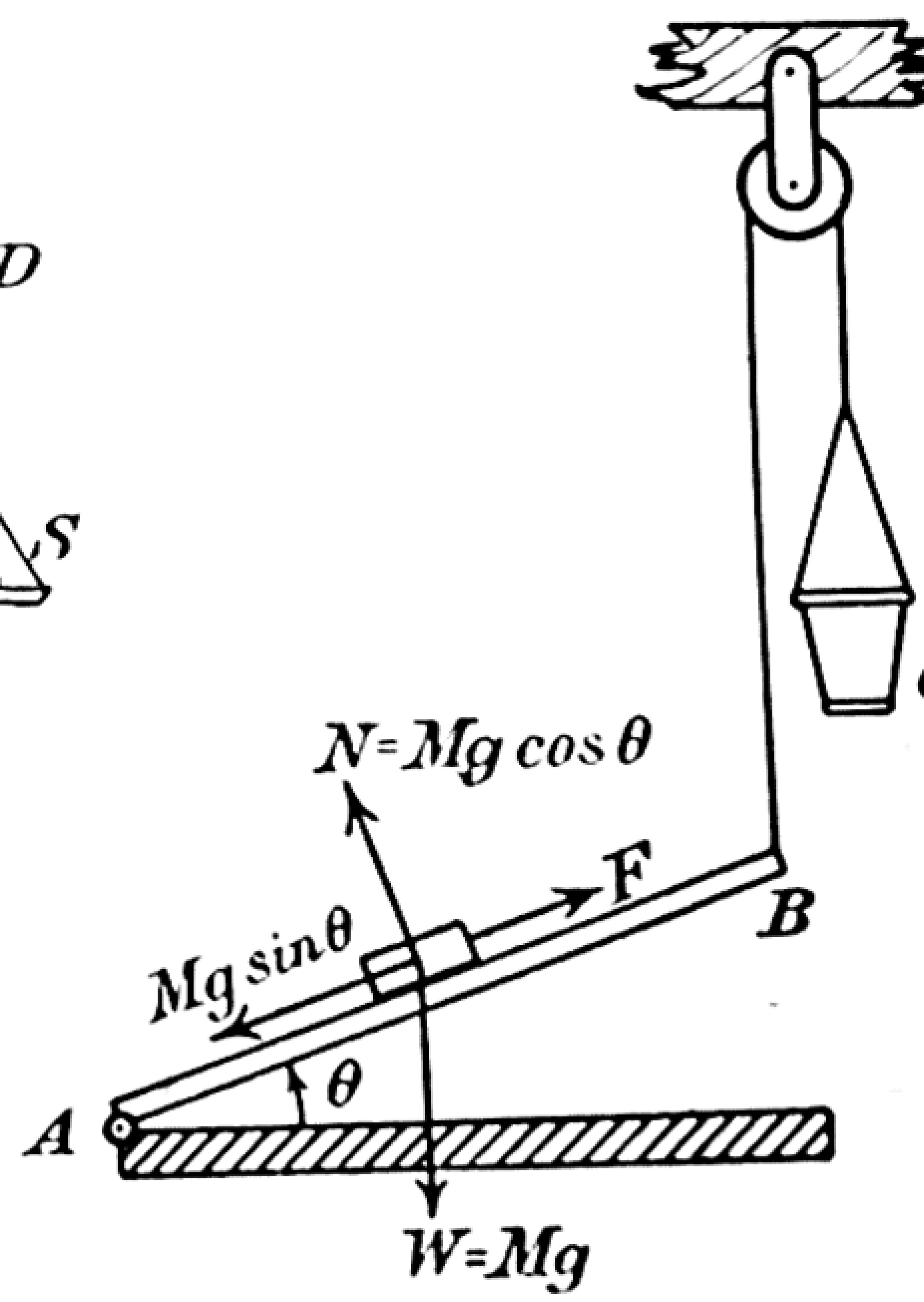


Fig. 3·19.

thrust between the surfaces of A and B in contact. Attached to the slider is a cord passing over a pulley D and carrying a scale-pan, S.

When there is no pull in the cord, the thrust of the board A upwards on the slider balances the weight of the latter and the system is at rest. When S is loaded there is a pull exerted in the string, but, provided that this is lower than a certain limit, no motion ensues. Forces exactly balancing the tension in the cord have been brought into play and resist the motion. These are the ***frictional forces***. It is found that up to a certain limit, in any given instance, just enough friction is called into play to prevent motion. If the pulling ceases, the forces due to friction also cease, for if they did not the body would move. Friction is

a self-adjusting force, for no more friction is called into play than is necessary to prevent motion. The amount of friction which may be exerted between two surfaces in contact is not, however, unlimited, for if the pull in the string is increased gradually a stage is finally reached when the body just begins to move; the friction is said to have reached its ***limiting value,*** and if the pull is further increased the slider is accelerated.

From experiments carried out on the lines suggested above, Coulomb established the following facts:—

(i) The limiting friction is independent of the area of contact between the surfaces so long as the thrust between them is unchanged.

(ii) The limiting frictional force, or limiting friction, is directly proportional to the normal thrust between the surfaces in contact, when the materials and nature of the surfaces remain unaltered, i.e. if F is the limiting value of the friction and N the normal reaction between two given surfaces, then the ratio $\frac{F}{N}$ is a constant. It is denoted by μ, and is termed the ***coefficient of limiting friction,*** or the ***coefficient of static friction.*** Hence

$$F = \mu N.$$

When the above slider is just about to move the forces acting on it are as shown in Fig. 3·18 (*b*), where R is the resultant of the normal reaction N and the friction F. The reaction R is inclined at an angle θ to the vertical, given by $\theta = \tan^{-1}\mu$. This angle is called the ***angle of friction.***

Experimental Determination of the Coefficient of Static Friction.—If the surface of the body under examination is flat, the coefficient of friction may be found as follows: The body is placed on a flat surface, AB, Fig. 3·19, pivoted about a horizontal joint at A. The other end, B, is attached to a bucket, C, into which lead shot may be poured to increase the tilt of the surface. Eventually a stage is reached when the body is just on the point of moving down the plane. Let θ be the inclination of the plane at this moment. The force acting down the plane is then $Mg \cdot \sin\theta$ which is equal and opposite to the frictional force, F, acting on the body. The normal reaction, N, the value for which is obtained by resolving forces in a direction normal to the plane, is $Mg\cos\theta$. We therefore have

$$\mu = \frac{F}{N} = \tan\theta.$$

The value of θ given by this equation is called the ***angle of repose.***

Kinetic Friction.—When slipping occurs between two bodies

in contact a frictional force continues to oppose the motion but, in general, the magnitude of this force is less than the frictional force existing just before slipping occurs. Experiment shows that as long as the motion is not too great, the frictional force F' is directly proportional to the normal reaction between the surfaces and is independent of the velocity, i.e.

$$F' = \nu N$$

where ν is the coefficient of kinetic friction.

Suppose that a body of mass m rests on a horizontal table which is not smooth. Then $N = mg$, and $F' = \nu mg$ when the body is moving. Suppose F_1 is the force applied to the body. Since F_1 and F' act in contrary senses, on a body of mass m, its acceleration a is given by

$$F_1 - F' = ma, \text{ or } a = \frac{F_1}{m} - \nu g.$$

In the absence of friction the acceleration would have been $\frac{F_1}{m}$, so that the effect of friction is to reduce the acceleration.

If the body is in motion and $F_1 < \nu mg$, a will be negative and the body will be brought to rest. To start the motion again a force greater than νmg will be required—it will be μmg.

Perry's Apparatus for determining the Coefficient of Kinetic Friction.—The essential parts of this apparatus are shown in Fig. 3·20 (*a*) and (*b*). A is a heavy wheel capable of rotation about a vertical axis,

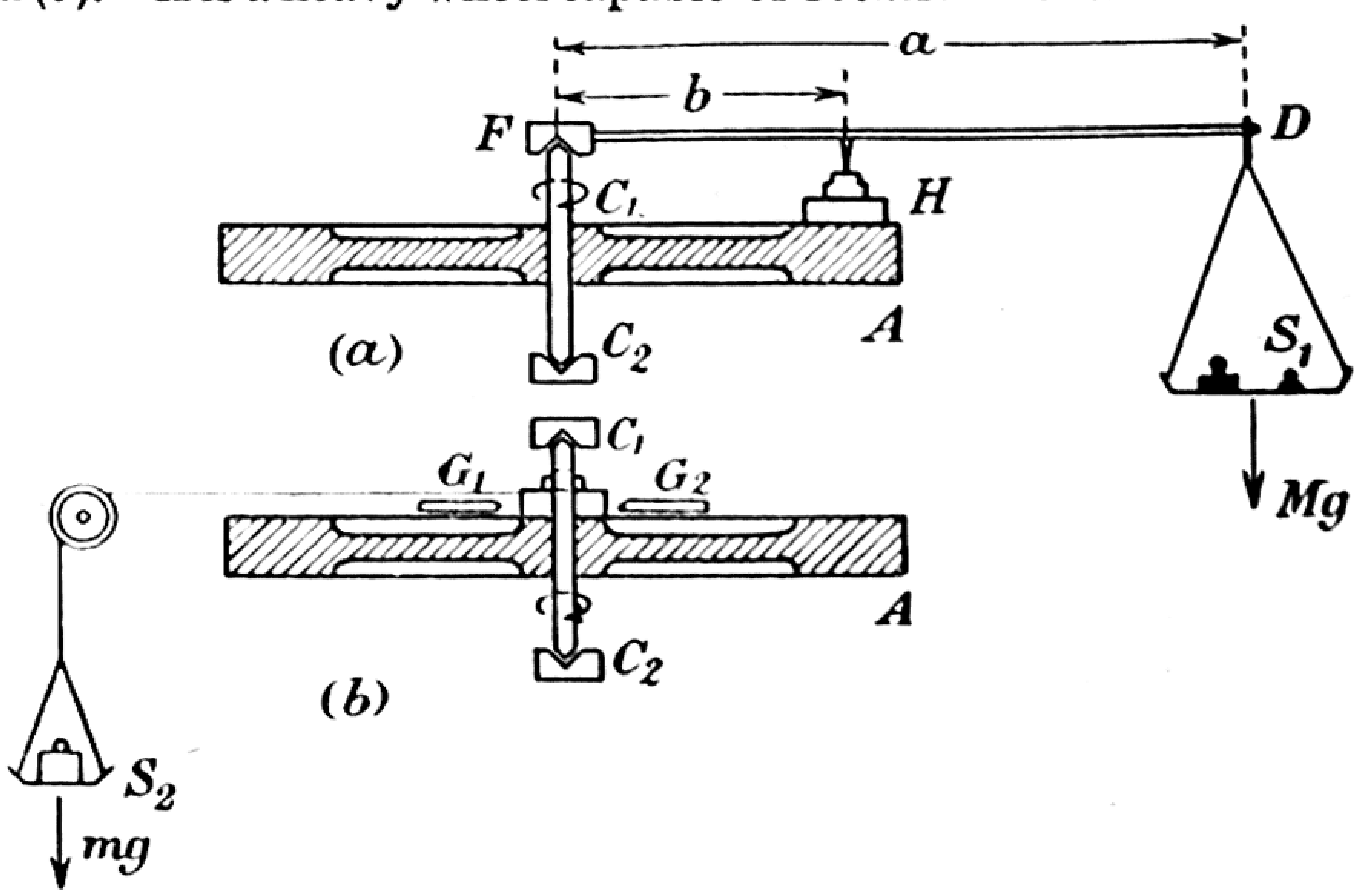


Fig. 3·20.—Perry's Apparatus for determining the Coefficient of Kinetic Friction between two Surfaces.

C_1C_2. DF is a lever carrying a scale pan, S_1, and having its fulcrum on the vertical axis C_1C_2. When the pan is loaded, as indicated, a thrust is exerted by the lever on a block, H—the surface of this block in contact with the wheel is flat. Attached to this block is a pan, S_2—the attachment is made by a cord passing over a pulley. The coefficient

of kinetic friction to be determined is that appropriate to the surface of H and that of the rotating wheel. When the system is stationary, H rests against a stop, G_1. The wheel is rotated so that H remains about half-way between the stops G_1 and G_2—it is said to be in 'floating equilibrium.' This condition is obtained by varying the load in S_2. Under these conditions the friction is equal to mg, the weight of the pan S_2 and its load. The normal reaction between the surfaces in contact is $Mg \cdot \frac{a}{b}$, where M is the mass of S_1 and the load in it. [We neglect the mass of the lever.]

$$\therefore \; \nu = \text{coefficient of kinetic friction} = \frac{m}{M} \cdot \frac{b}{a}.$$

Example. A body of mass 4 lb., hanging freely over the edge of a rough table, is connected by means of a light string passing over a smooth pulley at the edge, to a body of mass 2 lb. resting on the table. This is pulled 2 ft. along the table in 0·5 sec. from rest. What is the coefficient of friction ?

Let F poundals be the friction ; T poundals the tension in the cord. Then the resultant force pulling the 2 lb. mass is T − F, so that its acceleration is given by

$$T - F = 2a.$$

Using $s = \frac{1}{2}at^2$, we have $a = 16$ ft. sec.$^{-2}$

$$\therefore \; T - F = 32 \text{ poundals}.$$

Considering the 4 lb. mass, the resultant downward force acting on it is

$$128 - T = 4 \times 16.$$

$$\therefore \; F = 32 \text{ poundals} = 1 \text{ lb.-wt.}$$

$$\therefore \; \nu = \frac{F}{2g} = 0{\cdot}5.$$

The Friction Dynamometer.—The principle of this instrument, which is an application of the frictional forces existing between surfaces in contact to measure the rate at which work is done, is as follows: A large pulley wheel of radius r_1, Fig. 3·21, is rigidly fixed to the axle of the engine under test. A flexible belt having wooden blocks on its under side is placed over the outer rim of the wheel. One end of this belt carries a bucket W into which lead shot can be poured to increase its mass. The other end is fixed to a spiral spring attached to some rigid support [the floor]. Let r_2 be the outer radius of the belt. Suppose that the shaft makes n revolutions per second when the condition of 'floating equilibrium' has

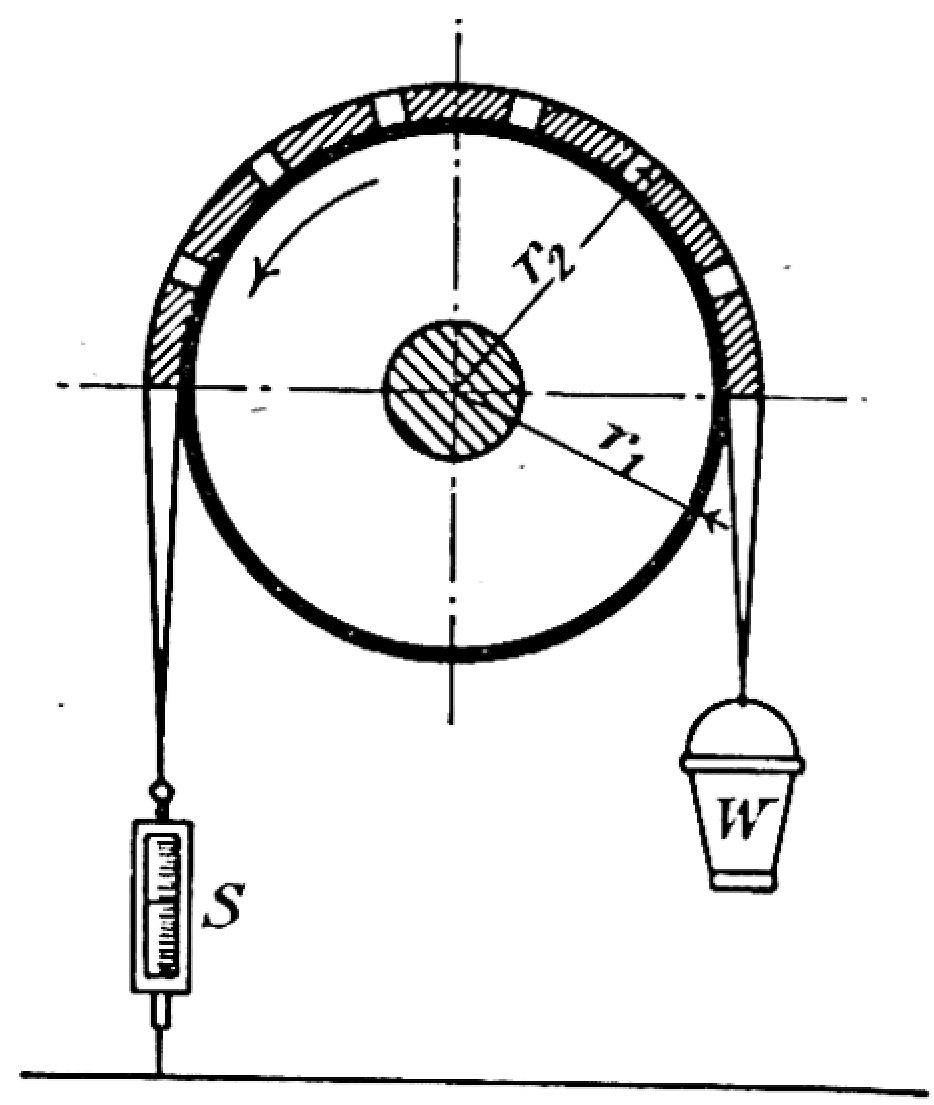


FIG. 3·21.—Friction Dynamometer.

[N.B.—The arrow on r_1 indicates that r_1 is the radius of the outer surface of the wheel.]

been obtained; let W be the weight of the bucket and its contents, while S is the reading on the spring balance. The moment about the axis of the shaft of the forces due to the weight W and the tension in the spring is $(W - S)\left(\frac{r_1 + r_2}{2}\right)$. This must be balanced by the moment of the frictional forces F about the same axis, viz. Fr_1. Now the distance through which the edge of the wheel moves against F is $2\pi r_1 n$ every second, so that the work done per second in overcoming friction is $2\pi n r_1 . F$. Eliminating F from this equation we find that the work done per second is $\pi n (r_1 + r_2)(W - S)$. This is the ***power*** of the engine, since it is the ***rate*** at which work is being done.

Rolling Friction.—To fix our ideas let us consider an engine wheel moving along a rail. There is never only contact at a point or even along a line normal to the rail, but always over a surface due to the elastic deformation of the bodies. Thus there is a small sliding motion of those parts of the surface in contact. There is thus brought into existence a frictional torque retarding the motion.

EXAMPLES III

1.—A force of 7 lb.-wt. acts on a mass of 29 lb. for 6 sec. How far has the body moved from rest? What is its final momentum?

2.—Find the resultant of two forces, 6 and 8 lb.-wt. respectively, acting on a body with an angle of $67\frac{1}{2}°$ between them. If this resultant acts on a mass of 1 cwt., determine the acceleration. Construct the velocity-time curve for the first 4 sec. of its motion.

3.—Enunciate the theorem known as the parallelogram of forces, and describe an experimental arrangement whereby this law may be verified.

4.—A circular disc has a radius of 10 cm. At a point 7 cm. from its centre a circular hole 4 cm. in diameter is punched. Calculate the position of the centre of gravity of the remaining metal.

5.—A uniform beam 18 ft. long, whose mass is 1 cwt., is inclined at 60° to the vertical. It is held in position by means of a horizontal cord 13·8 ft. from its lower extremity. Calculate the tension in the cord.

6.—What force is required to raise a load of 2 cwt. by means of the second system of pulleys if there are 4 pulleys in the lower block? A similar load is also raised by means of Weston's differential pulley in which $R = 1$ ft. and $r = 11$ in. Compare the velocity ratios in the two systems.

7.—Describe a balance, indicating the features which a good balance should possess.

8.—A body requires 20·61 gm. to hold it in equilibrium when placed in one pan of a balance, and 20·73 gm. when placed in the other. Calculate its true mass.

9.—Two masses, 5 and 12 lb. respectively, are attached to the ends of a uniform rod 6 ft. long, mass 3 lb. Where must a 20-lb. mass be

placed so that the whole will balance about a point 2 ft. 6 in. from the 5-lb. mass ?

10.—A mass of 10 gm. placed at the 97 cm. division on a uniform metre scale causes the whole to balance when the fulcrum is at the 55·3 cm. division. Calculate the mass of the scale.

11.—Two forces, 3·6 and 5·8 lb.-wt. respectively, have a resultant equal to 8·1 lb.-wt. What is the angle between the forces ? Check by a graphical method.

12.—Derive an expression for the time of oscillation of a simple pendulum. Explain how the intensity of gravity may be determined by means of such a pendulum.

13.—A uniform board ABC in the form of an equilateral triangle of 12 in. side weighs 3 lb. and has weights of 4 lb. and 5 lb. hanging from A and B respectively. Find a point from which the board may be suspended so that it sets in a vertical plane with AB horizontal and C pointing down. Is there more than one such point ? (L.S.C.)

14.—Explain, giving diagrams of the forces acting in each case, (*a*) how it is possible to sail a boat against the wind, (*b*) why the nose of a racing motor-boat rises out of the water, (*c*) why a railway ticket-collector leans backwards when alighting from a moving train.

15.—How would you compare accurately (*a*) the length of a standard yard with that of a standard metre, (*b*) the period of torsional oscillations of a horizontal rod suspended by a fine wire with that of a seconds pendulum ?

16.—A uniform cylinder of height h and radius r rests with its plane base on a rough inclined plane. The angle of inclination of the plane may be increased gradually from zero. Show that the cylinder will topple over before it slides if $2r/h$ is less than the tangent of the angle of friction.

17.—What is the radius of the sharpest bend which may be turned without skidding by a motor-car travelling at 30 ml. hr.$^{-1}$ on a level road if the coefficient of friction is 0·7.

18.—A body slides from rest down a rough plane in 5 sec. If the coefficient of friction is 0·42, and the inclination of the plane 25°, what is the length of the plane ?

19.—The distance between the scale-pan knife-edges in a balance is 30 cm. The central knife-edge is at a perpendicular distance of 1 cm. above the middle point of the line joining the scale-pan knife-edges. The centre of gravity of the beam is 2 cm. below the central knife-edge. The mass of the beam is 850 gm. ; that of each scale-pan 100 gm. Find the deflection of the beam when masses of 50 and 51 gm. are placed in the pans.

20.—Explain the construction of a good beam-balance, pointing out the factors which determine (*a*) its accuracy, (*b*) its sensitiveness.

How could you find the mass of a body if you had to use a balance which was not true ?

CHAPTER IV

THE ELEMENTS OF HYDROSTATICS

Density and Specific Gravity.—The density of a substance is defined as the mass of the substance per unit volume. ***A priori*** this statement calls for little comment, for whether 1 cm.^{3} or 1000 cm.^{3} are used in the experimental determination the same value for the density is obtained within the limits of experimental error. If, however, one adopts the modern view that all substances consist of molecules or atoms which are not in contact with one another, and which do not fill the whole of available space, some further remarks are necessary. Suppose that some imaginary being is free to move in and out amongst the molecules; his idea of the density of the medium will be very different from ours, for the particular volume which he chooses may contain many or a few such molecules, or even none at all. These statements are made here to show the student that some of our most commonplace ideas, i.e. ideas gained from a macroscopic view of things, are very different when the structure of matter is considered microscopically.

The idea of density is frequently confused with that of ***specific gravity,*** which is defined as the ***ratio*** of the mass of a given substance to that of an equal volume of water at the same temperature. Since this value is a ratio it is independent of the system of units used in the experimental determination, whereas the density, being a mass per unit volume, must always be expressed in gm. cm.$^{-3}$, or lb. ft.$^{-3}$, etc.

Fluids.—Solids are those substances which offer a considerable resistance to any force endeavouring to change their size or shape. On the other hand fluids, such as alcohol or nitrogen, cannot offer any permanent resistance to impressed forces tending to alter their shape. The term ***fluid*** is used to include both ***liquids*** and ***gases,*** the fundamental difference between liquids and gases being that the latter always occupy the whole of the space which is available, whereas liquids are always characterized by the presence of a free surface. This free surface is horizontal for such masses of liquid as are found in pools, etc., but becomes curved when the mass of

liquid is larger, as in the case of a sea ; in both instances the surface is everywhere perpendicular to the earth's radius at that point, but it is only in the second that the curvature can be detected easily. The thrust on any solid surface in contact with a fluid at rest is everywhere normal, i.e. perpendicular to the surface. If this were not so the thrust could be resolved into forces perpendicular and parallel to the surface, and this parallel force would cause motion of the body.

Pressure.—Whenever a force, F, is applied to an area, s, so that it is distributed equally and acts normally to the surface, then $\frac{F}{s}$, the force per unit area, is termed the ***pressure*** on that area. If the force is not distributed equally we may determine the pressure at any point on that area by constructing a small area round the said point. If δF is the force acting on such a

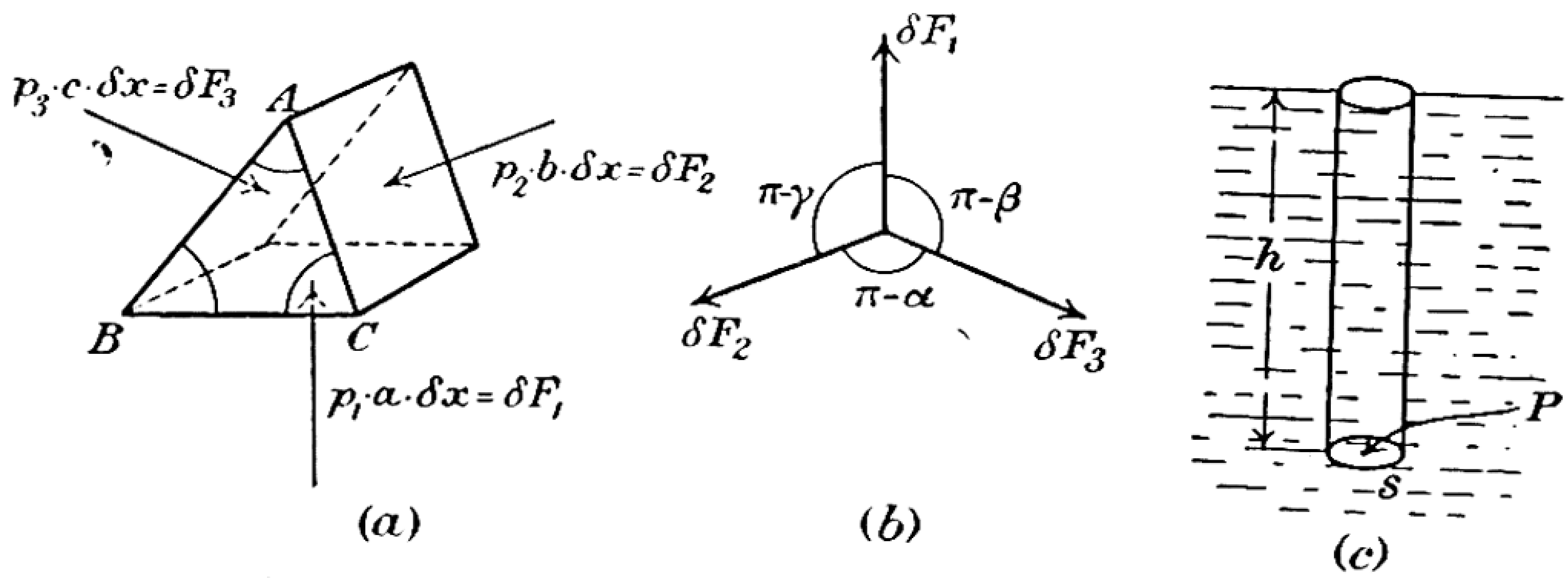


FIG. 4·1.—Pressure at a point in a Liquid at rest.

small area δs, then when δs is sufficiently small we may consider the force to be distributed uniformly over δs, so that the pressure is $\frac{\delta F}{\delta s}$; in the limit this becomes $\frac{dF}{ds}$.* [In the c.g.s. system the absolute unit of pressure is one dyne. cm.$^{-2}$; in the f.p.s. system it is one poundal. ft^{-2}. The corresponding gravitational units are the gm.-wt. cm.$^{-2}$, and the lb.-wt. ft.$^{-2}$]

To Show that the Pressure at a Point in a Fluid at rest is the same in all directions.—Consider any point in the fluid, and suppose that a wedge in the form of a triangular prism of arbitrary section ABC, Fig. 4·1 (a), surrounds the point. Then the fluid inside the wedge is in equilibrium under the action of

(i) its weight acting vertically downwards,

(ii) the thrusts on its faces.

If the wedge is very small the weight of the fluid in it, depending on the product of three small quantities, is negligible in comparison

* $\frac{dF}{ds} = \lim\limits_{\delta s \to 0} \frac{\delta F}{\delta s}$.

with the forces acting on the sides, each of which depends on the product of two small quantities. Now the forces acting normally on the two ends of the prism are equal and opposite so that they may be omitted in the problem before us. Let the forces acting normally on the three other faces be δF_1, δF_2, and δF_3: these must be in equilibrium since the fluid is at rest. If the angles of the section are α, β, and γ, the angles between the lines of action of the forces are $(\pi - \alpha)$, $(\pi - \beta)$, and $(\pi - \gamma)$ respectively—cf. Fig. 4·1 (b). Then

$$\frac{\delta F_1}{\sin(\pi - \alpha)} = \frac{\delta F_2}{\sin(\pi - \beta)} = \frac{\delta F_3}{\sin(\pi - \gamma)}.$$

But $$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}, \qquad [a, b, \text{ and } c \text{ are the sides of } \triangle ABC.]$$

$$\therefore \frac{\delta F_1}{a\,.\,\delta x} = \frac{\delta F_2}{b\,.\,\delta x} = \frac{\delta F_3}{c\,.\,\delta x},$$

where δx is the length of the wedge, i.e. the pressures over the faces of the prism are equal.

Pressure at a Point in a Fluid.—To determine the pressure at a point distant h below the free surface of a liquid which is at rest, and whose density is ρ, we imagine a small horizontal area s drawn round P and consider the liquid contained in the right cylinder having this area as its base—cf. Fig. 4·1 (c). This cylinder of liquid is in equilibrium under (a) the upthrust on the base, (b) its own weight, (c) the thrusts due to the pressure of the surrounding liquid on its sides. Since these are everywhere normal to the surface they have no vertical component, so that for equilibrium the weight, W, of the cylinder of liquid must be equal to the upthrust on the base.

Now, using the c.g.s. system of units,

W (dynes) = weight of a column of liquid of height h (cm.), and cross-sectional area s (cm.^{2}),
= mass of this column $\times g$, the acceleration due to gravity,
= [volume of this liquid, v, (cm.^{3}) $\times$ its density, ρ, (gm. cm.$^{-3}$)] $\times g$,
= $(sh\rho)g$ (dyne).

Hence, F, the total thrust on the area s is $(sh\rho)g$ (dyne). The pressure P at any point in the base is therefore given by

$$P = \frac{F}{s} = (g\rho h)\ (\text{dyne. cm.}^{-2}).$$

From the above we see that the pressures at two points in the same horizontal plane in a liquid at rest must be equal. This may

be shown by cutting a piece of brass tubing at right angles to its axis and arranging the two new ends thus formed in the same horizontal plane. To facilitate this adjustment a flat sheet of metal and a spirit level may be used. A beaker containing liquid is then placed so that the ends of the brass tubes are immersed. When the liquid is at rest the ends of the tubes must be at the same depth below its surface. If the two tubes are connected together by means of a T-piece and rubber tubing, bubbles of gas appear from the two ends at the same time when pressure is applied to the open end of the T-piece. In carrying out this experiment narrow tubes must not be used since other forces become appreciable so that the simplicity of the experiment is lost: the reason for this will be noticed later [cf. p. 119].

Archimedes' Principle.—When a body is immersed either wholly or partly in a fluid at rest, it displaces a volume of fluid equal to that of the immersed portion, and experiences an upthrust due to the liquid displaced; the magnitude of this upthrust is equal to the weight of the displaced fluid. Let A, Fig. 4·2, be such a body. If the body is supposed to have been removed, and the space it occupied filled with some of the fluid, the forces arising from the superincumbent fluid are unaltered. Now the resultant of these forces just balances the gravitational force acting on this mass of the fluid, viz. its weight—the above resultant must act vertically upwards. When the body was in the fluid these forces were still existent and must therefore have reduced the effect of the earth's attraction on the body, i.e. its weight was apparently diminished by an amount equal to the weight of fluid displaced. If the body A were suspended from a balance this apparent loss in weight would be detected as an apparent loss in mass.

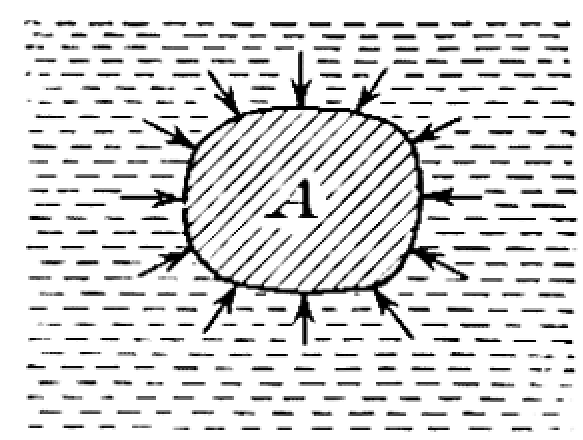


FIG. 4·2.

A similar argument holds when the body is only partly immersed in the fluid.

Experimental Verification of Archimedes' Principle.—The apparatus commonly employed to demonstrate the truth of the above principle is shown in Fig. 4·3. It consists of two cylinders A and B which are of such dimensions that the solid cylinder B just slides into A and fills it completely. When in this position the whole is suspended from the arm of a balance and the balance equilibrated, [sand may be used]. B is then withdrawn and suspended in a beaker containing liquid from below A with the aid of the hooks provided. The equilibrium of the balance is thereby destroyed, but it may be restored by pouring some of the same liquid into A as that in which B is immersed. Equilibrium

will be established when A is completely filled with liquid. This verifies that the upthrust on B when this is completely immersed in a liquid is equal to the weight of the liquid displaced by B.

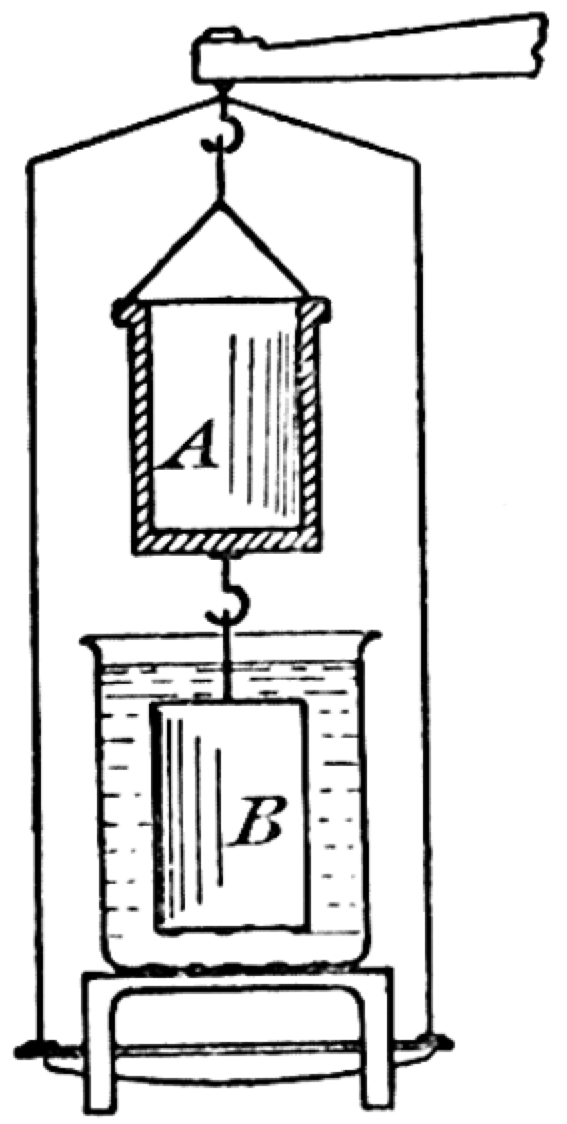


FIG. 4·3.— Apparatus to verify Archimedes' principle.

The experiment should also be repeated when B is a closed hollow cylinder such that it floats in the liquid. The procedure is exactly as above except that a piece of lead sufficient to cause the cylinder to sink is suspended from A and immersed in the liquid throughout the whole time that the experiment is being performed.

In order to vary this second part of the experiment the cylinder B is made of iron and mercury is the liquid used. A piece of tungsten, density 18·4 gm. cm.$^{-3}$, will be required to sink the iron. Brass or copper must not be employed in place of iron since these metals form amalgams with mercury.

The Principle of Flotation.—If the resultant of the forces acting on a body partly submerged in a liquid at rest and due to the liquid exactly balances the weight of the body, then that body floats. But it has been seen above, that the resultant of these forces is equal to the weight of the liquid displaced, so that for a floating body it may be said that the weight of the liquid displaced is equal to the weight of the body.

Experimental Methods for the Determination of Density.—(*a*) The density of a solid substance insoluble in water can be found by determining its mass in air and then in water. The apparent loss in mass of the solid is equal to the mass of the water displaced, and since 1 cm.^{3} of water has a mass of 1 gm. the volume of water displaced and hence the volume of the solid are known. For accurate work account must be taken of the fact that it is only when the temperature of the water is 4° C. and the external pressure 1 atmosphere that 1 cm.^{3} of water has a mass of 1 gm. If an experiment were made at 20° C. how could the volume of the solid be found? From tables it is known that the density of water at 20° C. is 0·998 gm. cm^{-3}. Suppose that the apparent loss in mass of the submerged body is 13·61 gm. Then the volume is equal to the mass divided by the density, viz.

$$\frac{13{\cdot}61}{0{\cdot}998} = 13{\cdot}63 \text{ cm.}^3$$

(*b*) If the body floats in water it must be caused to sink by using a heavy piece of metal called a ***sinker***. Let m_1 be the mass of the

body in air, m_2 the mass of the body in air plus the sinker in water, and m_3 the mass when both are suspended together in water. Now

$m_2 = m_1 +$ apparent mass of sinker in water,
$\quad = m_1 +$ mass of sinker in air $-$ mass of water displaced by sinker, and
$m_3 =$ apparent mass of both in water,
$\quad = m_1 -$ mass of water displaced by the solid $+$ mass of sinker in air $-$ mass of water displaced by sinker,
$\quad = m_2 -$ mass of water displaced by solid.
$\therefore\ m_2 - m_3 =$ mass of water displaced by the solid.

If ρ_0 is the density of water at the temperature of the experiment, the volume of the water displaced is $(m_2 - m_3) \div \rho_0$: this is the volume of the solid. The density of the solid is therefore

$$\frac{m_1\rho_0}{(m_2 - m_3)}.$$

(*c*) The density or specific gravity bottle is a small glass container fitted with a ground glass stopper. A capillary hole in this stopper permits an excess of liquid to be removed and at the same time ensures a constant volume for the bottle. It is filled with the liquid and then cleaned and filled with distilled water, the mass of liquid in each instance being determined. The specific gravity of the fluid is the ratio of these masses ; the density is easily calculated at any temperature as in (*a*). In using the bottle care must be taken to see that no air bubbles remain clinging to the sides of the bottle, and that the bottle has been completely filled at the same temperature in both instances.

(*d*) The density of a solid, available as a powder or as small crystals, may be determined with the aid of a density bottle. The method will be illustrated by considering how to determine the density of some crystals (e.g. sugar, copper sulphate, etc.) which are soluble in water but not in some other liquid (e.g. turpentine). The following observations must be made.

Mass of bottle	$= m_1$
,, ,, ,, + crystals	$= m_2$
,, ,, ,, + crystals and turpentine to fill	$= m_3$
,, ,, ,, + turpentine to fill	$= m_4$
,, ,, ,, + water to fill	$= m_5$
$\therefore$ Mass of solid used . . .	$= (m_2 - m_1)$
,, ,, turpentine required to fill bottle when crystals are present . . .	$= (m_3 - m_2)$
Now ,, ,, turpentine to fill bottle .	$= (m_4 - m_1)$
$\therefore$,, ,, turpentine, the volume of which is equal to that of the crystals. . .	$= (m_4 - m_1) - (m_3 - m_2$

To find the volume of this mass of turpentine its density must be known. But the mass of water, of density ρ_0, required to fill the bottle is $(m_5 - m_1)$. The density of the turpentine is therefore

$$\frac{m_4 - m_1}{m_5 - m_1} \cdot \rho_0.$$

$$\therefore \text{ Volume of crystals } = \frac{(m_4 - m_1) - (m_3 - m_2)}{(m_4 - m_1)\rho_0} \cdot (m_5 - m_1).$$

$$\therefore \text{ Density of crystals } = \frac{(m_2 - m_1)\ (m_4 - m_1)\rho_0}{[(m_4 - m_1) - (m_3 - m_2)](m_5 - m_1)}.$$

(*e*) If the liquid whose density is required is only available in small quantities then its density may be found as follows:—A uniform glass capillary tube of suitable diameter (say 1 mm.) is selected, cleaned, dried, and its mass determined. A long length of mercury is placed in the tube, preferably by attaching a small piece of rubber to the tube, placing a bubble of mercury in the rubber and applying pressure at the open end of the rubber tube. This operation has a filtering action upon the mercury and enables the mercury to be introduced without undue contamination of the tube which is the result if suction is applied by the mouth. The mass of the mercury is determined. The tube is then filled with liquid and its mass found. In either case it is necessary to measure the length of the fluid in the tube. If very accurate results are required corrections to this length must be made owing to the existence of curved surfaces at the ends of the column. As a first approximation one adds (or subtracts) a length equal to two-thirds the diameter [1] of the tube, if the lengths have been measured as the distances between the extreme points at which the mercury (or liquid) is in contact with the glass. From the mass m, and corrected length l, of the mercury the mean radius of the tube is found, for if ρ is the density of the mercury at the temperature of the experiment, the volume of mercury is $\frac{m}{\rho}$ and this equals $\pi r^2 l$ so that

$$r = \sqrt{\frac{m}{\pi\rho l}}.$$

If r is small, m will also be small. It is then better to introduce in turn several pellets of mercury, measure the length of each, and determine their total mass, $\Sigma(m)$, say. Let this be μ—the only mass which has to be determined. If $\Sigma(l)$ is the total length of all the pellets,

$$r = \sqrt{\frac{\Sigma(m)}{\pi\rho\Sigma(l)}} = \sqrt{\frac{\mu}{\pi\rho\Sigma(l)}}.$$

[1] An approximate value of the diameter is obtained by finding a wire which will fit the tube and measuring its diameter with a screw gauge.

If M is the mass of a liquid whose density D is required, and this occupies a length L of the above tube, then

$$\frac{M}{D} = \pi r^2 L,$$

$$\text{or } D = \frac{M}{\pi r^2 L}, \text{ where } r \text{ is now known.}$$

In the above it was stated that the tube should be uniform in cross-section. This is only essential if the lengths of the mercury pellet and the column of liquid introduced are not equal, but a non-uniform tube may be used if the lengths of mercury and liquid columns are equal, for the tube may then be used as a density bottle of known volume.

The Internal Radii of Tubes.—The last paragraph has shown us how the internal radius of a narrow tube may be found, but the same method cannot be extended to wider tubes since the mercury would not fill the entire cross-section of the tube. We therefore proceed as follows :—A cork is inserted at one end of the tube and a little water (or mercury, if greater precision is desired) added so that when the tube is vertical the surface of the water is at some fiducial mark A. The mass of the whole is found. More water is then introduced until the level is at a second fiducial mark B. The mass is again determined and from the observations the volume of the tube between the marks A and B deduced. By proceeding in this way any errors due to the shape of the cork are avoided. If the length AB is known, the radius of the tube can be calculated. This same method may be used to find the radius of a test-tube. It should be noticed that this method, like the one above, only determines the *mean* radius of the tube.

Hydrometers.—Two of the usual forms of hydrometer, which is an instrument used to determine the density of liquids or solids, are shown in Fig. 4·4; the first consists of a bulb A, at the lower extremity of which there is a small bulb B, containing mercury or lead shot. The neck between A and B is solid so that the mercury cannot be displaced. In the pattern shown here the bulb B is part of a mercury thermometer the scale of which is placed inside A. This enables the temperature of the liquid to be

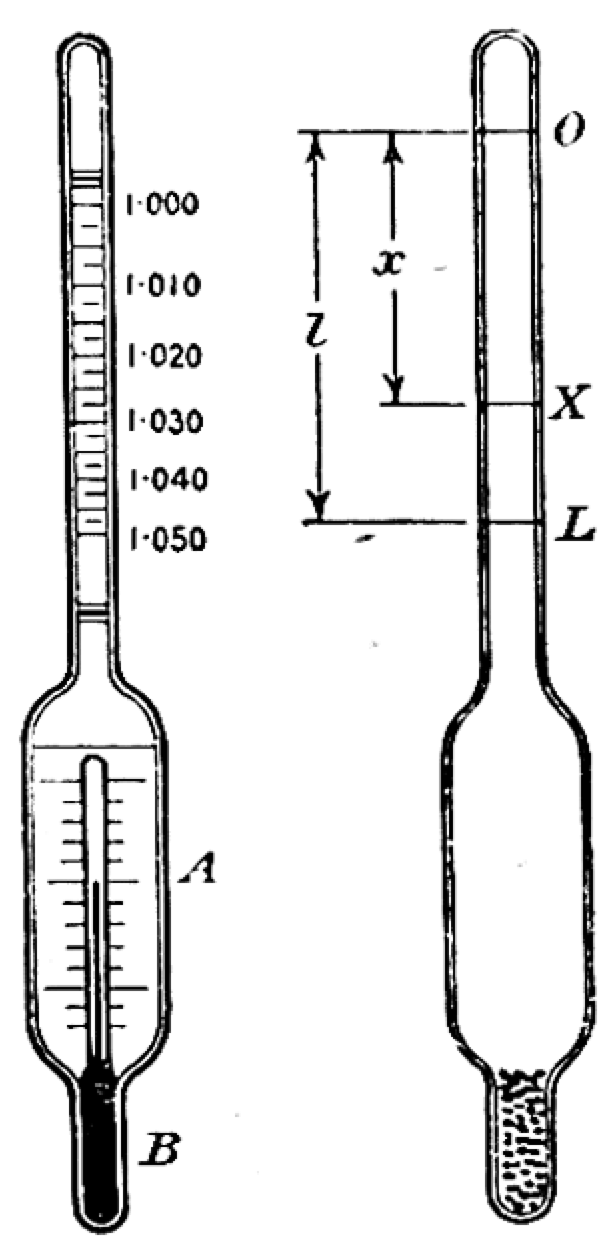


FIG. 4·4.

(a) A Common or Constant Mass Hydrometer.
(b) Theory of a Floating Hydrometer.

observed without using a second thermometer. To the other extremity of A there is attached a long narrow tube which carries the scale of the instrument. The scale numbers generally refer to density, and the scale is so situated that the number at the point where the stem emerges from the fluid in which the hydrometer is immersed gives the density of the fluid.

The Equilibrium of a Floating Hydrometer (Elementary Theory).—Let us assume that the hydrometer is designed for use with liquids whose densities are greater than that of water. Let O, Fig. 4·4 (*b*), be the zero mark.

Let m = mass of hydrometer.

V = volume of the hydrometer up to its zero mark,

v = volume per unit length of the stem.

Then mg is the weight of the hydrometer, and this is the gravitational force pulling it downwards. Suppose that when the instrument floats in a liquid of density ρ the increase in length of the emergent part of the stem is n. Since the total volume of liquid displaced is $(V - nv)$, the upthrust of the liquid on the hydrometer is $(V - nv)\rho g$. For equilibrium

$$mg = (V - nv)\rho g,$$

i.e.

$$m = (V - nv)\rho.$$

The Graduation of a Common Hydrometer.—To calibrate this instrument, assuming that the stem is uniform in cross-section, one may proceed as follows. Suppose that O is the mark to which the instrument sinks when it is floated in water of density ρ_0 gm. cm.$^{-3}$; let L, Fig. 4·4 (*b*), be the mark when the hydrometer floats in a liquid of density ρ_1, this density being known or determinable. Let l be the distance OL. Let X be the mark on the stem to which the instrument sinks when floating in a liquid of density ρ. Call OX = x. The problem before us is to determine x in terms of l, ρ_1, and ρ: we then give values to ρ numerically equal to 1·00, 1·01, 1·02, etc., and so find out where these graduations must be placed.

If V is the volume of the instrument up to the mark O, and v the volume per unit length of the stem, we have, by the principle of flotation,

$$\begin{aligned} V \times \rho_0 &= \text{mass of water displaced} = \text{mass of hydrometer} \\ &= \text{mass of liquid displaced} \\ &= (V - lv)\,.\,\rho_1 = (V - xv)\rho. \end{aligned}$$

Hence

$$xv = \mathrm{V}\left(1 - \frac{\rho_0}{\rho}\right), \text{ and } lv = \mathrm{V}\left(1 - \frac{\rho_0}{\rho_1}\right),$$

so that

$$x = l\left[\frac{1 - \frac{\rho_0}{\rho}}{1 - \frac{\rho_0}{\rho_1}}\right].$$

In all accurate work with hydrometers it is very essential that the liquid surface should be clean. The following experiment verifies the above statement. A deep glass vessel is thoroughly cleaned and provided with a side tube near its base so that it may be completely filled with tap-water. A hydrometer is placed therein and the water allowed to overflow continuously. In this way a very clear water surface is obtained. The flow of water is stopped and the equilibrium position of the hydrometer noted. The water surface is then touched with a rod which has been wetted in a soap solution : this contaminates the water surface and the hydrometer rises—probably one or two millimetres. This is because the surface tension [cf. p. 113] of the liquid has been reduced and the hydrometer is not pulled down to the same extent as when the surface tension of the water had its maximum value, i.e. as when its surface was clean.

Nicholson's Hydrometer.—This instrument, which was designed for determining (*a*) the densities of solids and (*b*) those of liquids whose densities do not differ very much from that of water, consists of a hollow vessel, A, comprising a cylinder and two conical portions—cf. Fig. 4·5. The instrument carries upper and lower pans, B and C, respectively ; C is loaded with lead shot so that the hydrometer floats in an upright position when placed in a liquid. The hydrometer is made of brass and nickel-plated so that the tendency for air bubbles to cling to it shall be minimized. To find the density of a liquid the instrument is first placed in the liquid and masses added to the upper pan until a definite mark on the stem just touches the surface. It is generally somewhat difficult to judge this coincidence exactly so that it is better to solder a bent pin, P, to the stem of the hydrometer and always bring the point of the pin into contact with the surface of the liquid. This coincidence is best ascertained by looking at the reflexion of the pin in the surface of the liquid from a point below. If m_1 is the mass in the upper pan and M is the mass of the instrument itself, then, according to the principle of flotation, the mass

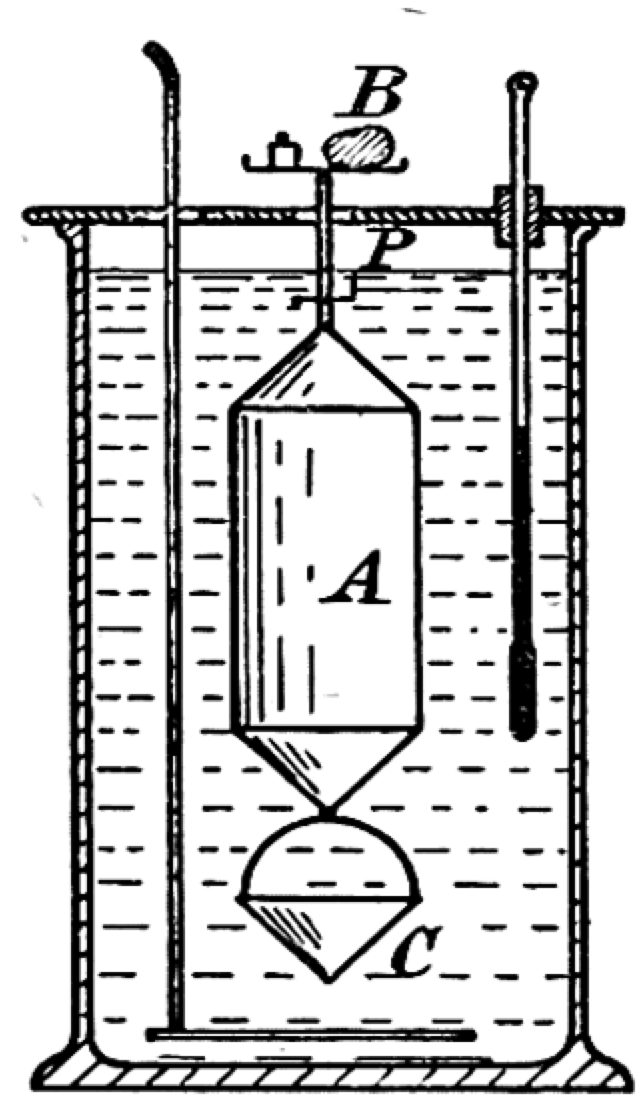


FIG. 4·5.—Nicholson's Hydrometer.

of the liquid displaced is $M + m_1$. The hydrometer is then washed and floated in water when a mass m_2 will be required in the upper pan in order to sink the instrument to P. The mass of the water displaced is $M + m_2$. If ρ_0 is the density of water at the temperature of the experiment the volume of water displaced is $(M + m_2)/\rho_0$. This is equal to the volume of liquid displaced. The density of the liquid is

$$\frac{(M + m_1)\rho_0}{(M + m_2)}.$$

If the instrument is floated in a liquid whose density differs considerably from unity, there is a tendency for it to tilt. This may be avoided by placing a suitable piece of brass in the lower pan during this part of the experiment, and making a correction as follows. Let m_1 = mass in the upper pan required to sink the instrument to the mark P when the piece of brass of mass μ and density ρ_1 is placed in the lower pan. Then the mass of the liquid displaced by the hydrometer is $M + m_1 + \mu$, and the volume of the liquid displaced, being the volume of liquid displaced by the hydrometer alone plus the volume of the piece of brass, is $\left(\frac{M + m_2}{\rho_0}\right) + \frac{\mu}{\rho_1}$. The density required is therefore

$$\frac{(M + m_1 + \mu)}{\left[\frac{M + m_2}{\rho_0} + \frac{\mu}{\rho_1}\right]}.$$

If a hydrometer is properly used, reliable results are obtained even for liquids whose densities do not differ much from unity,

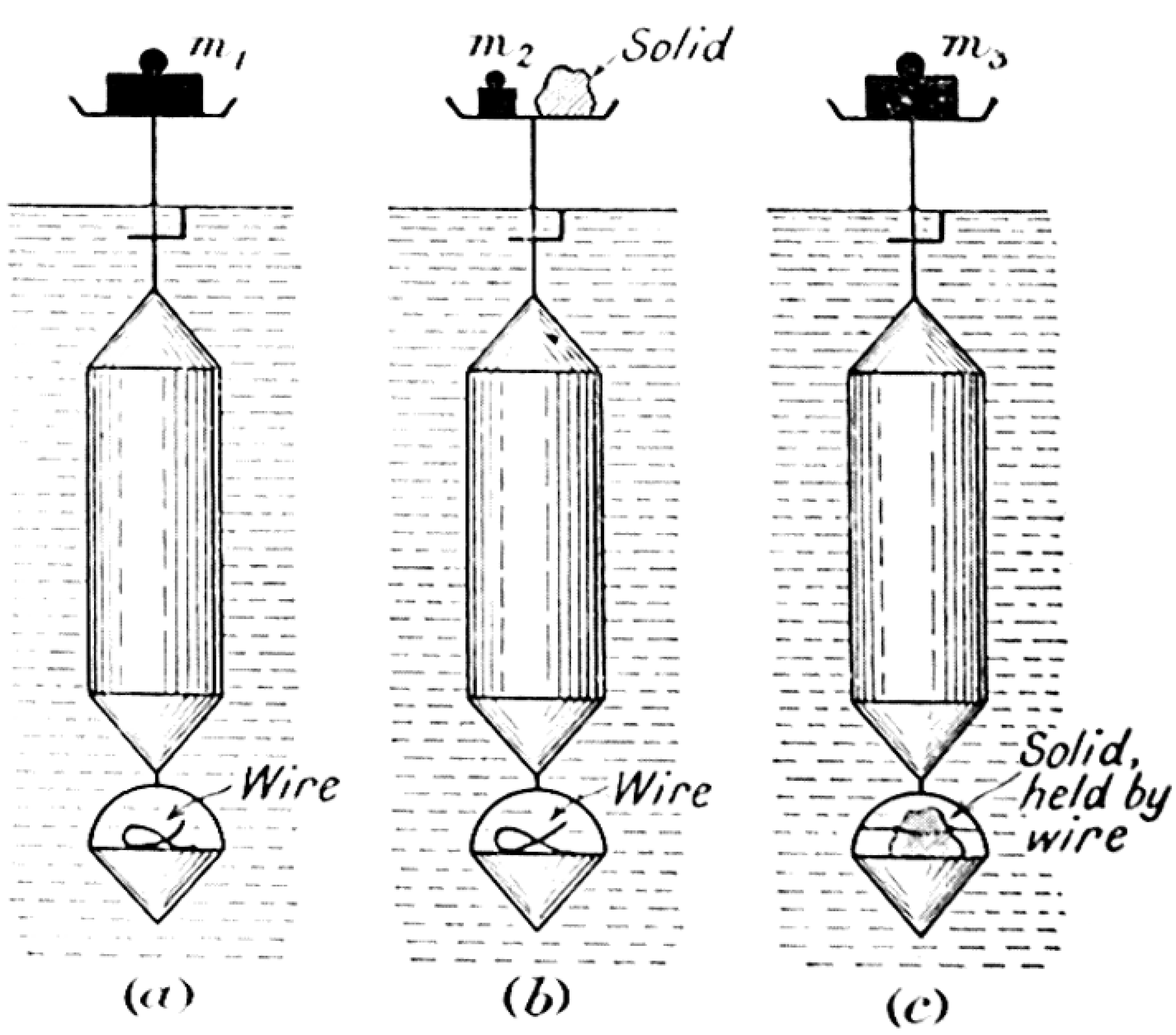


Fig. 4·6.—Principle of the Nicholson (or Constant Immersion) Hydrometer.

since the masses of large volumes of liquid and water have to be determined.

To determine the density of a solid the hydrometer is first floated in water as before and the mass required to sink the hydrometer to P ascertained. Let this be m_1, cf. Fig. 4·6 (*a*). The solid is then placed on the upper pan and the mass necessary to sink the instrument to the same mark again found, cf. Fig. 4·6 (*b*). Let this be m_2, so that the mass of the solid in air is $(m_1 - m_2)$. The solid is then placed on the lower pan when it will be found that a mass m_3 is necessary to sink the hydrometer to the same fiducial mark, cf. Fig. 4·6 (*c*). This mass will be greater than m_2 due to the upthrust of the water on the solid. Now by the principle of flotation, the mass of the water displaced in each instance is equal to the mass of the floating object. Hence, considering the state of affairs indicated in Fig. 4·6 (*c*), we have

Mass of water displaced by hydrometer when floating as in (*a*) + mass of water displaced by solid
$$= \text{M} + m_3 + \text{mass of solid in air.}$$
$\therefore$ (M + m_1) + mass of water displaced by solid
$$= \text{M} + m_3 + m_1 - m_2.$$
$\therefore$ Mass of water displaced by solid
$$= (m_3 - m_2).$$

If the density of water is ρ_0, the volume of the solid is $\left(\dfrac{m_3 - m_2}{\rho_0}\right)$, so that its density is $\left(\dfrac{m_1 - m_2}{m_3 - m_2}\right)\rho_0$.

It will be noticed that this method applies equally well to solids which float, the only difference being that the solid must be tied to the lower pan. This may be done with the aid of a piece of wire and if this is allowed to remain on the lower pan throughout the experiment its mass need not be known.

Alcoholometry.—The term alcoholometry is applied to the determination of the strength of spirits. In the days of the alchemists rough-and-ready means were used. A piece of cloth was moistened with the spirit and a light applied: ignition indicated strong spirit. Sometimes an oil was poured upon the surface of the spirit; strong spirit floated on the surface of the oil. Later the spirit to be tested was used to moisten gunpowder—when a light was applied rapid combustion indicated a strong spirit; steady burning indicated a spirit which was regarded as 'good, rightfull and of vertue' and was known as 'proof' spirit. In 1666 some friction arose between importers of French brandy and the customs officials concerning the rate of duty chargeable on the liquid. There were two rates, 4*d*. and 8*d*. per gallon, for liquors of different qualities, and the revenue officials, guided by the sense of taste, asked for the higher rate. The decision was contested by the importers, but was eventually ratified; the

test was made statutory in 1670. Fraudulent merchants, however, attempted to disguise the taste of their brandies, and so other means had to be found. BOYLE first thought of using a hydrometer for testing spirits, and after various improvements it has become the standard instrument for such purposes.

'Over' and 'Under' Proof.—The term 'proof' is applied to spirits having a density 0·91976 gm. cm.$^{-3}$ at 15·56° C. (60° F.); this corresponds to 49·28 per cent. of alcohol by weight or 57·10 per cent. alcohol by volume. If the ***over***-proof strength is added to 100, the sum represents the number of volumes of spirit at proof strength which that particular over-proof strength would make. Thus, 100 vol. of spirit at 16° over-proof are equivalent to 116 vol. of proof spirit, whereas 100 vols. of 16° under-proof are equivalent to 84 vol. of proof spirit. Absolute alcohol is 75·35° over-proof.

Sike's Hydrometer.—This is the particular form of instrument used in alcoholometry. It consists of a gilded brass bulb, 1·5 in. in diameter, to the bottom of which is fixed a counterpoise. The stem is a thin rectangular strip graduated in arbitrary units. Tables are supplied which convert readings into terms of over- or under-proof strengths.

Stability of Floating Bodies.—The principle of flotation [cf. p. 74] asserts that the mass of the floating object is equal to the

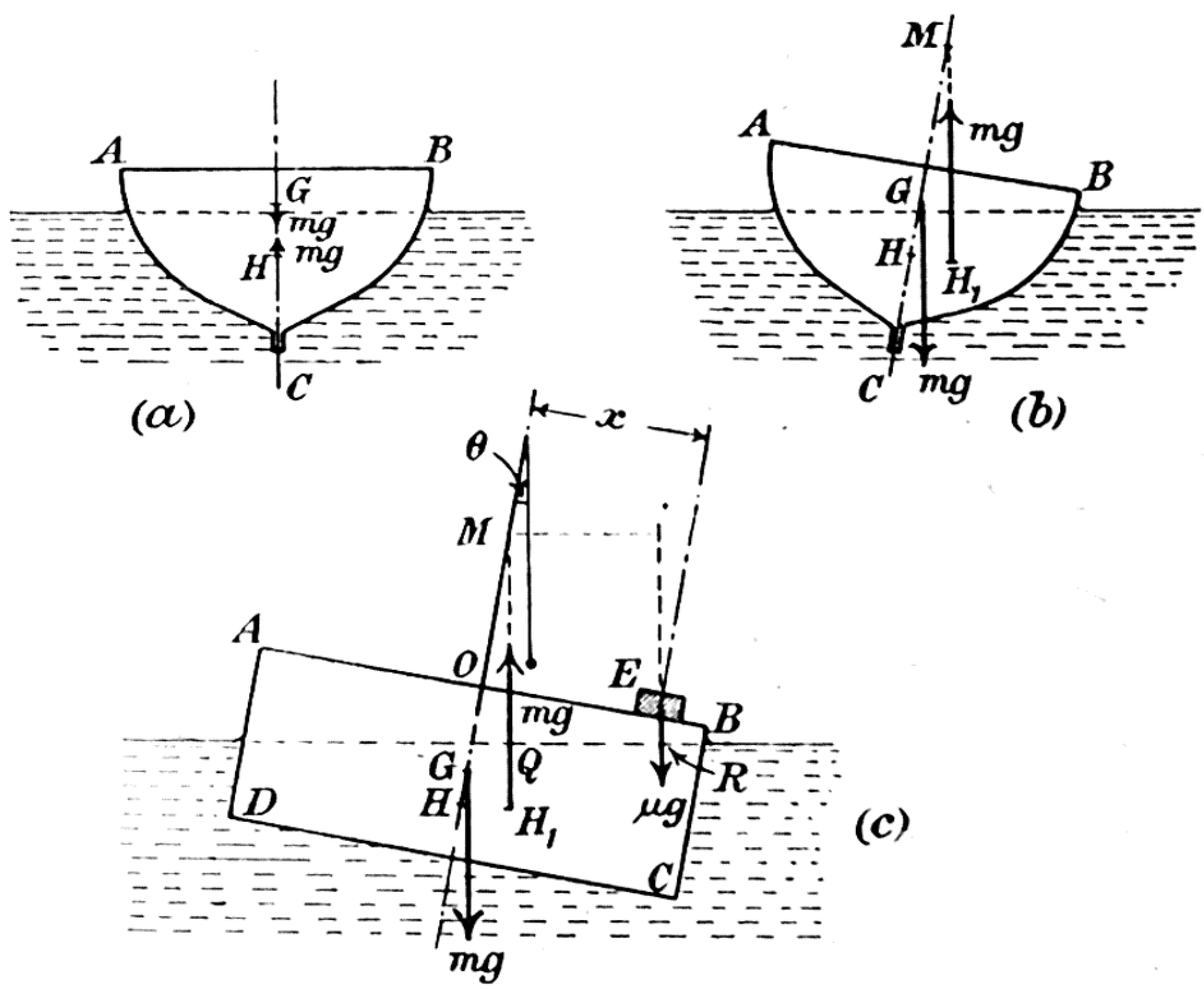


FIG. 4·7.—The Metacentric Height of a Floating Body.

mass of the liquid displaced. This condition alone is not sufficient to determine the equilibrium of the floating object. If it is in

equilibrium the weight of the solid must not only be equal to the upthrust of the liquid displaced, but these two forces must act in the same straight line.

Now while these two conditions are sufficient to determine the equilibrium of the floating body, the stability of that equilibrium requires further discussion.

Consider a floating body (e.g. a ship) in the position of equilibrium. If m is the mass of the ship, it is in equilibrium under the action of two forces, its weight, mg, where g is the acceleration due to gravity, acting vertically downwards through G, Fig. 4·7 (a), the centre of gravity of the body, and the upthrust, also mg, acting vertically upwards through H, the centre of gravity of the displaced liquid. The point H is termed the centre of buoyancy. Now HG is vertical when the ship is in its equilibrium position. We shall assume that this line is marked on the ship and that it moves with it when the equilibrium is disturbed. When the ship is displaced through a small angle, let the centre of buoyancy move to a position H_1, Fig. 4·7 (b), in the plane of the diagram. The mass of displaced liquid will remain unaltered, but its resultant upthrust will now act vertically through H_1. If this line of action of the upthrust cuts HG produced in M, then M is the ***metacentre*** of the ship, while the distance GM is the ***metacentric height*** of the ship.

The ship is now acted upon by a couple and if M is above G this couple will tend to restore the ship to its equilibrium position, i.e. the equilibrium is stable. Unstable equilibrium follows when M is below G.

To Determine Experimentally the Metacentric Height of a Rectangular Piece of Wood Floating in Water.—Consider that rectangular section ABCD, Fig. 4·7 (c), of the floating body which passes through G, the centre of gravity of the body. Suppose that the body is displaced through a small angle θ by placing a body of mass μ at E. Let M be the metacentre whose position is to be determined experimentally, and suppose that GM cuts AB in O. Let OE $= x$. If m is the mass of the wood, and μ is small compared with m, so that the mass of the displaced liquid may be considered constant, the three forces maintaining the body in equilibrium are its weight mg, acting vertically downwards through G, the upthrust mg acting vertically upwards at H_1, the centre of buoyancy in the disturbed position of the wood, and the weight μg of the mass at E which acts vertically downwards. By taking moments of forces about Q, the point of intersection of the water line with H_1M, we obtain GM, for

$$mg \,.\, \mathrm{GM} \,.\, \sin\theta = \mu g \,.\, \mathrm{RQ},$$

where R is the projection of E on the water line. If θ is small, $\sin\theta = \theta$, and $RQ = x\cos\theta = x$.

Hence
$$GM = \frac{\mu x}{m\theta}.$$

The angle θ is deduced from observations on the position of a plumb-line attached to the wood as indicated.

Pressure of the Atmosphere.—The earth is surrounded by an envelope of mixed gases consisting of oxygen and nitrogen for the main part, but also containing carbon-dioxide, water vapour, and in smaller amounts argon, neon, krypton and xenon. This mixture is a fluid and, as such, exerts a pressure. In general, this pressure diminishes with increasing altitude, and is such that at distances greater than 50 miles above the earth's surface, the air is so rarefied as to be almost non-existent. Fig. 4·8 shows the effect of placing a tube, completely filled with mercury, in a reservoir of this substance. Whether the tube is inclined or not, the vertical height of the column, providing the mercury does not fill the tube entirely, is the same in each tube and is a measure of the pressure of the atmosphere under the prevailing conditions. The vacuum above the upper surface of the mercury is called a ***Torricellian*** vacuum, and should contain only traces of mercury vapour. This space is so called because it was discovered in 1643 by an Italian named TORRICELLI. Such tubes are the essential part of all mercury barometers.

FIG. 4·8.

The Fortin Barometer.—The distinctive feature of this instrument, Fig. 4·9, is the device used for keeping the level of the mercury in the reservoir constant. This permits the use of a fixed scale—generally engraved on the brass case, A, surrounding the barometer tube, B. The reservoir bottom, C, is made of chamois leather and is moved by means of a plunger, the motion being imparted by the rotation of the screw S; this is moved so that the mercury level in the reservoir is coincident with the extremity of an ivory point P, whenever observations are being made. The tip of P coincides with the zero of the scale on A. The above coincidence is examined by viewing the reflexion of the point in the mercury surface. To determine the position of the upper surface of the mercury on the scale of A, the tube E, sliding inside A and operated by the milled knob D, is adjusted so that its lower end is level with the mercury surface. A vernier scale on E enables the position of the mercury

surface to be determined. After some months' use air tends to find its way along the glass-mercury surface; this is prevented from reaching the vacuum by means of the re-entrant glass joint X. The glass tube used in such a barometer is shown in Fig. 4·9 (*b*).

Boyle's Law.—Gases are fundamentally different from solids and liquids. The fact that a given mass of gas is at a certain temperature does not define its volume definitely, for a gas always occupies the whole of the available space in the vessel enclosing it. If the volume of the gas is increased the gas still fills the whole of the vessel, but the pressure it exerts on its walls is reduced. Similarly, if the volume is decreased, the pressure is increased. BOYLE, in 1662, investigated the relationship between the volume of a given mass of gas and the pressure to which it is subjected, and his results are expressed by the law which bears his name: '***The volume of a given mass of gas at constant temperature is inversely proportional to the pressure to which it is subjected.***'

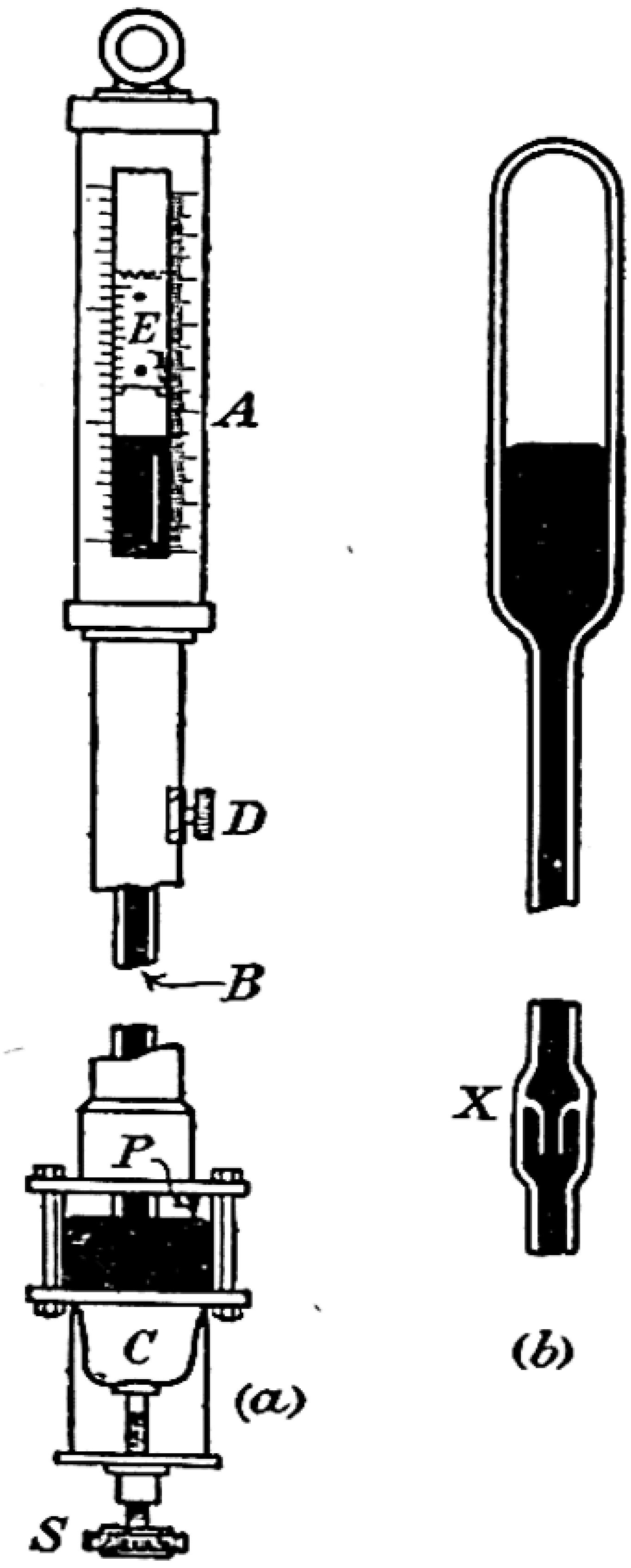


FIG. 4·9.—A Fortin Barometer.

Experimental Verification of Boyle's Law.—Fig. 4·10 (*a*) is a diagrammatic representation of the essential parts of the apparatus. It consists of a burette or other suitably calibrated vessel, A, connected by means of thick rubber tubing to a wide tube, B, containing mercury. C is a two-way tap leading either to a tube D, containing calcium chloride, or to a tube E. At the top of D there is a rubber bung through which pass E and another tube F which may be closed by a small glass cap and piece of rubber tubing. A loosely packed plug of glass wool, G, at the lower end of D prevents particles of the chloride from entering A. The tap C is first placed so that connection is made between A and E [cf. Fig. 4·10 (*c*)]. When B is raised the air or other gas in A is expelled into D via the tube E. During this operation care should be exercised to prevent the mercury from coming into contact with the grease on the tap, for mercury is easily contaminated. C is then rotated so that there is direct connection between D and A [cf. Fig. 4·10 (*b*)]. When B is lowered dry gas enters A. This operation is repeated several times so that the gas finally left in A is dry. With the tap C closed, B is raised to a considerable

height. If the mercury level in A continues to change, the tap C is leaking, so that this defect must be remedied before proceeding.

When it has been shown that the apparatus is free from leaks the volume of gas in A is noted, and the levels of the mercury in A and B are observed by means of the scale S. The difference between these two observations is a measure of the pressure difference between that in A and that of the atmosphere. If the barometric height is observed, the pressure of the gas in A in terms of cm. of mercury at room temperature may be deduced. A series of observations with the pressure in A both greater and then less than atmospheric is made. If now a graph is drawn showing the relation between p, the pressure, and V, the volume of the gas, a curve is obtained, but its nature cannot be directly inferred. But since it is expected that the observations will support the relationship $p \propto \frac{1}{V}$, i.e. $pV = \kappa$, where κ is a constant, we should plot log p and log V. If the points lie on a straight line whose slope is -1, the validity of Boyle's law over the range of pressures investigated will have been established, for $\log p + \log V = \log \kappa$ = constant, is the equation to a straight line whose slope is -1. The validity may also be tested by plotting p against $\frac{1}{V}$, when a straight line should be obtained. Its slope is κ.

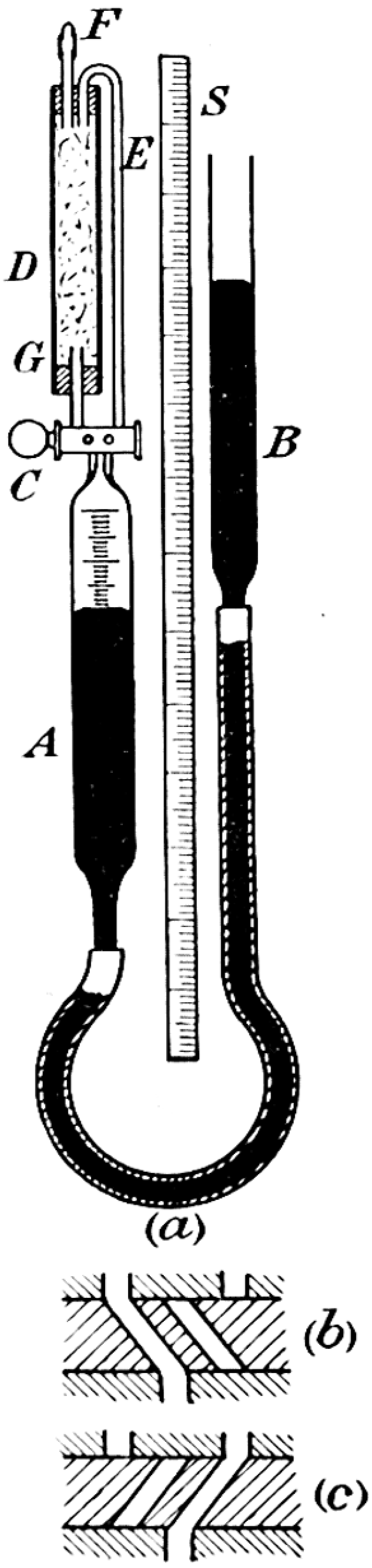


Fig. 4·10.—Boyle's Law Apparatus.

The actual method used by Boyle (1662) to establish his law for air was to observe the volume of air in the closed limb of a U-tube at atmospheric pressure and then at different pressures. He assumed the law to be valid and calculated what the volume should be for the pressures applied. This calculated volume was compared with the observed volume and the agreement was found to be very good.

[The numbers in Ex. 26, p. 101, have been taken from Boyle's original paper.]

Experiment. Clean and dry a glass tube about 40 cm. long and 0·3 cm. in diameter. Introduce a pellet of mercury about 10 cm. long into the tube. Observe the barometric height. Determine the

length of the tube occupied by the enclosed air when the tube is vertical and also when the tube is rotated through 180° in a vertical plane, i.e. when the pressure of the enclosed gas is greater and then less than atmospheric by an amount depending on the length of the mercury pellet. Introduce other pellets into the tube and repeat the observations. Hence investigate the validity of Boyle's law.

Hare's Density Apparatus.—This apparatus enables us to compare the densities of two liquids, so that if the density of one is known, that of the other may be deduced. It consists of two vertical tubes, AB and CD, Fig. 4·11. The upper ends of these tubes are connected to a T-piece and stop-cock, E: their lower ends each dip into one of the liquids under examination. By applying suction at E the liquids may be brought to convenient positions in the tubes. Let us suppose that these positions are P_1 and Q_1 respectively. If D and d are the densities of the two liquids while H_1 and h_1 are equal to the heights of P_1 and Q_1 above the exposed surfaces of the liquids, the difference in pressure between the inside and outside of the apparatus is $g\mathrm{DH}_1$ or gdh_1, i.e. $\frac{d}{\mathrm{D}}=\frac{\mathrm{H}_1}{h_1}$. In actual practice it is at least inconvenient, and certainly undesirable, to adjust the ends of the scales S and T so that they are in contact with the exposed surfaces of the liquids. To avoid this, a long pin (or screw) is pushed through a piece of wood resting on top of the containing vessel in each instance, the pins being vertical, and their positions adjusted until their lower ends just touch the liquid surfaces. S and T are then used to measure the heights of P and Q above the tops of the pins, and if the lengths of the pins are known, H_1 and h_1 are easily deduced.

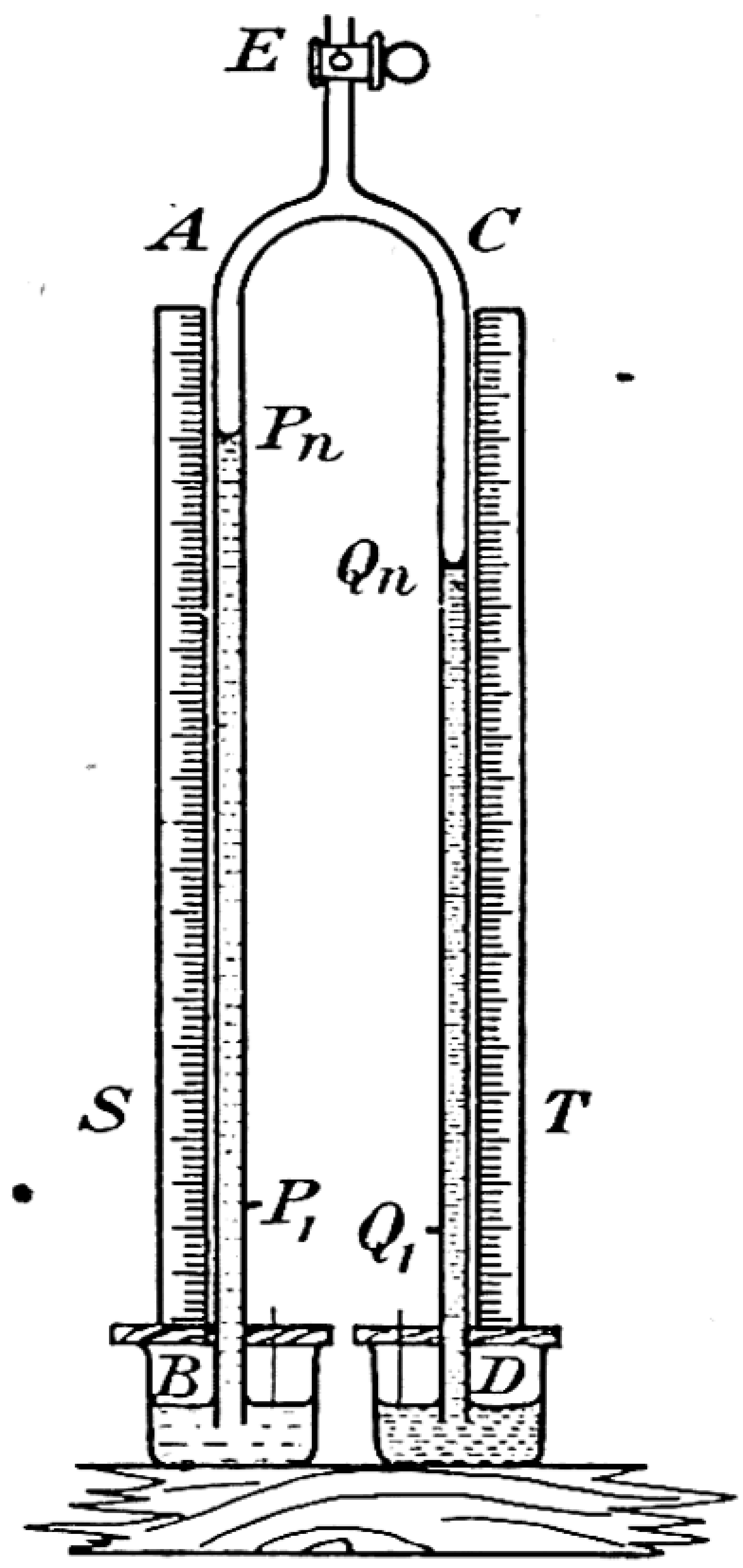


FIG. 4·11.—Hare's Apparatus.

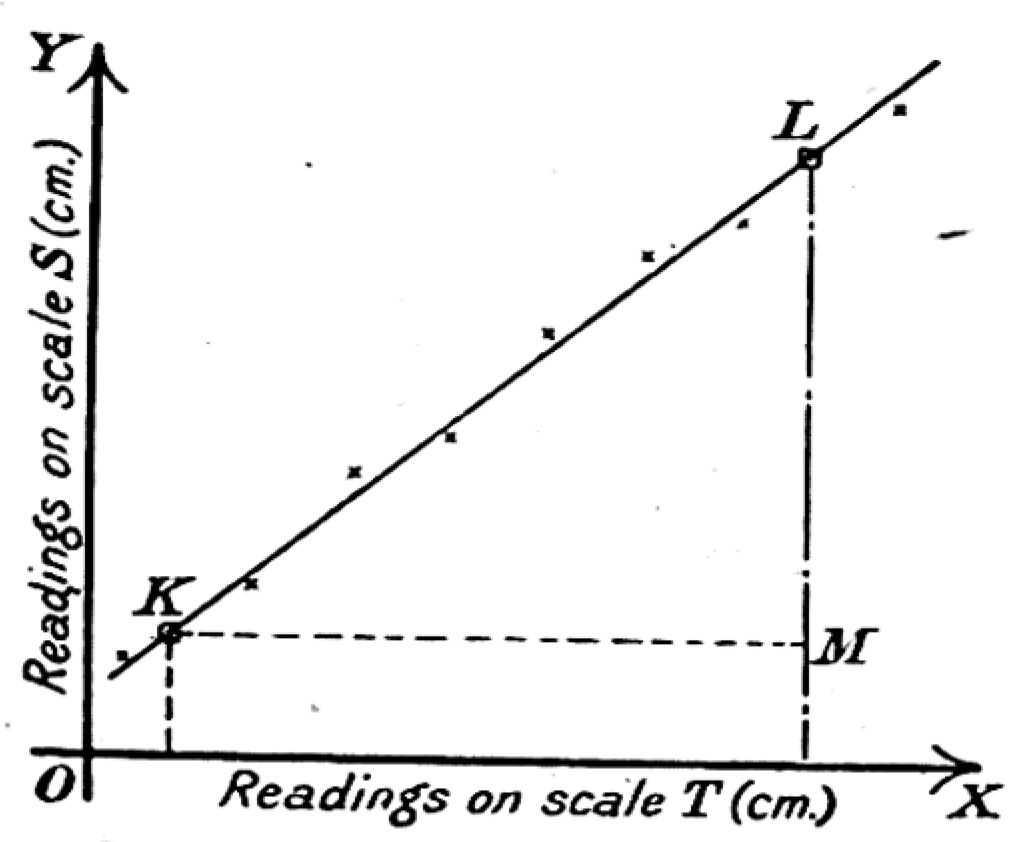


FIG. 4·12.

A series of observations with the levels of the liquids at different positions in the tubes is made, care being taken to see

that the tubes are thoroughly wetted. The observations are then plotted as in Fig. 4·12 and the best straight line drawn. Let K and L be two points on this line and draw KM and LM parallel to the axes of reference. It follows that ML and KM will be proportional to the same change of pressure inside the apparatus, so that if we denote them by H and h respectively, $g\text{DH} = gdh$, i.e. $\frac{d}{\text{D}} = \frac{\text{H}}{h}$. Generally the liquid in AB is water so that in the c.g.s. system of units $\text{D} = 1$ gm. cm.$^{-3}$, and therefore $d = \frac{\text{H}}{h}$ gm. cm.$^{-3}$.

Buoyancy in Gases.—It has already been shown that any solid immersed in a liquid experiences an upthrust equal to the weight of the liquid displaced. Gases, too, exert an upthrust on bodies in them equal to the weight of the gas displaced. This may be demonstrated in the following manner. A, Fig. 4·13, is a

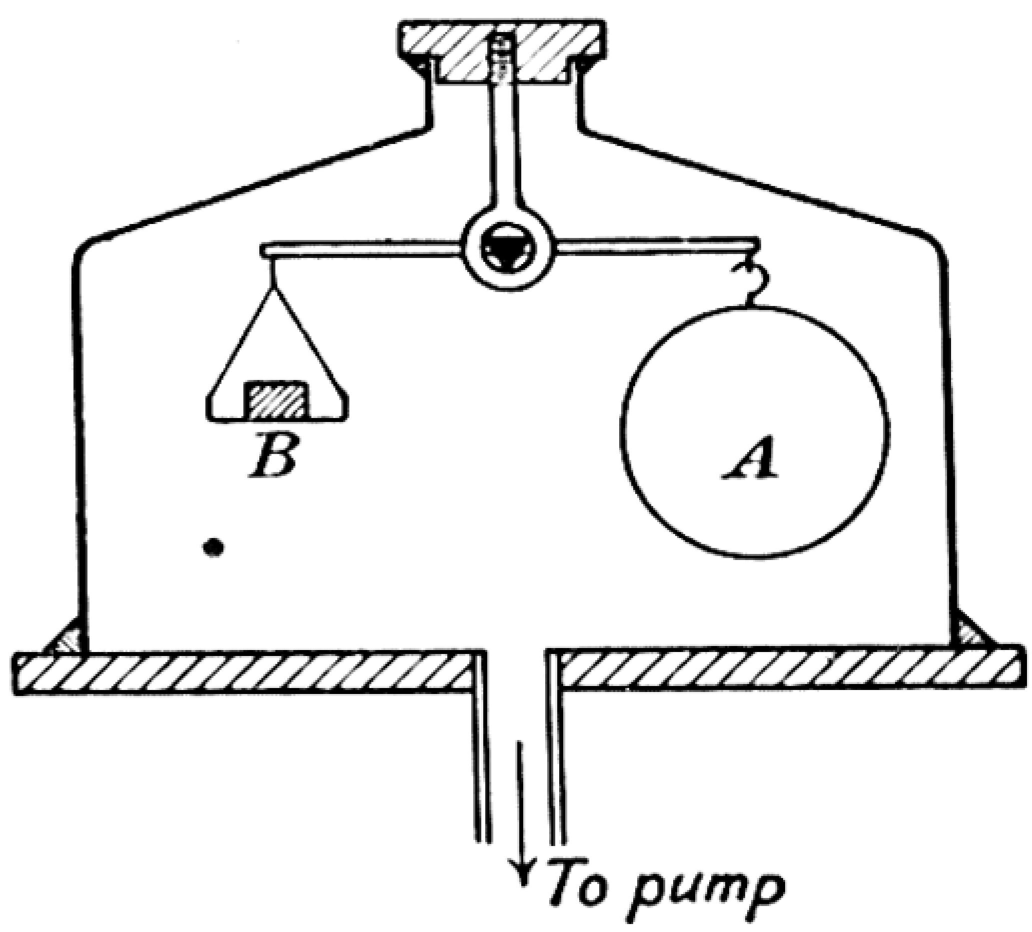


FIG. 4·13.—Buoyancy in Gases.

hermetically sealed vessel—a glass globe, for example—suspended from one arm of a balance and counterpoised by a mass, B. The whole is placed inside a large bell-jar which may be exhausted. As the air is removed from the jar the up-thrust on the large body A is much reduced in comparison with that on the counterpoise B. In consequence, the equilibrium of the balance is destroyed and A falls.

Correction for Buoyancy in Determining the Mass of an Object.—Let m be the mass of the weights (brass) necessary to counterpoise a given object, the weighing operation being carried out in air. Let ρ_1 be the density of brass, ρ_2 that of the material of the solid whose mass is being determined, and ρ_a that of the air under existing conditions. Let M be the true mass of the solid.

Then its volume is M/ρ_2, so that the upthrust on it due to the air displaced is

$$\left(\frac{M}{\rho_2}\right)\rho_a \cdot g.$$

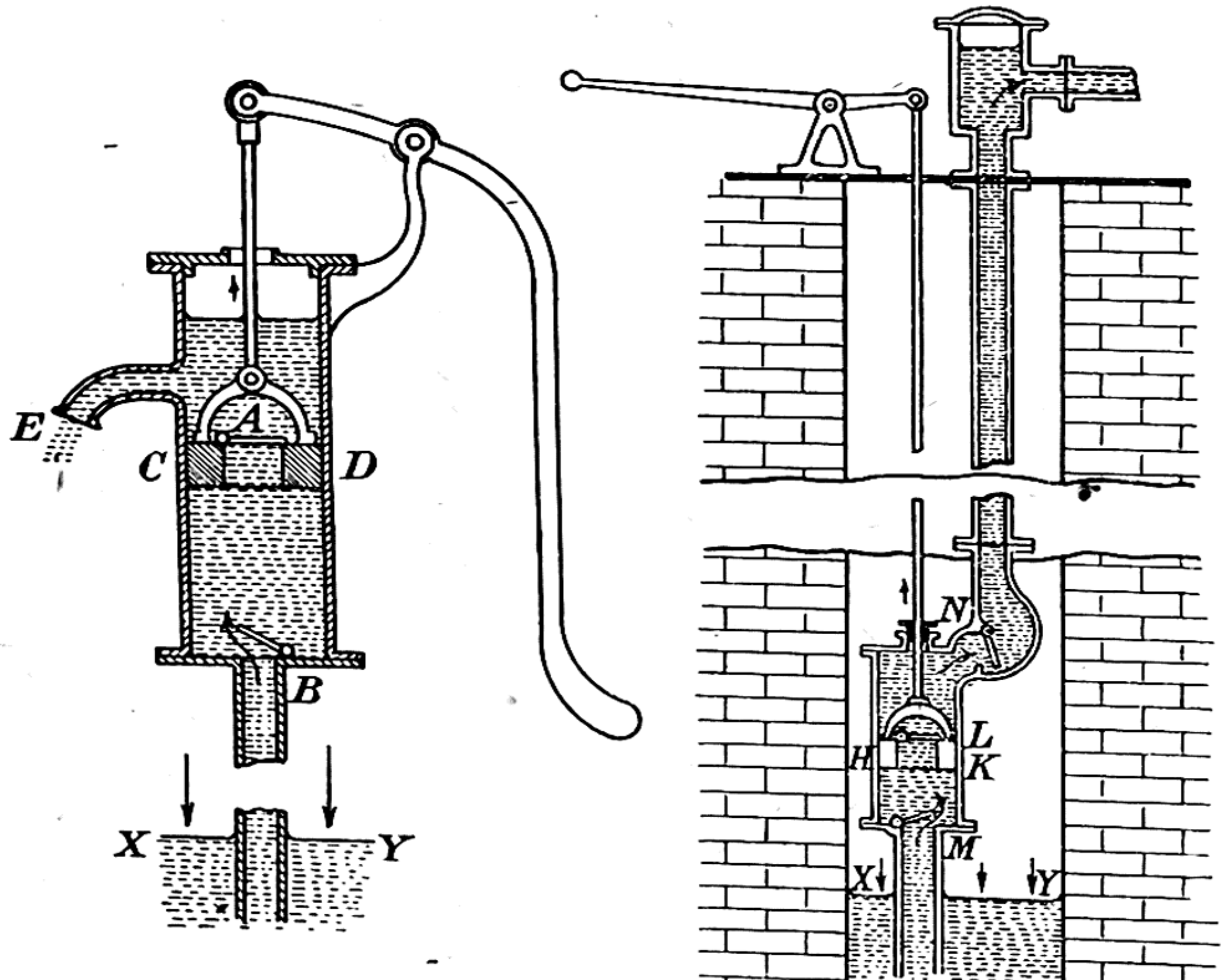
On the brass weights the upthrust is $\left(\dfrac{m}{\rho_1}\right)\rho_a \cdot g$. For equilibrium

$$Mg - \left(\frac{M}{\rho_2}\right)\rho_a g = mg - \left(\frac{m}{\rho_1}\right)\rho_a g$$

i.e.
$$M\left[1 - \frac{\rho_a}{\rho_2}\right] = m\left[1 - \frac{\rho_a}{\rho_1}\right]$$

$$\therefore\ M = m\left[1 - \rho_a\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\right], \text{ since } \rho_a \text{ is small.}$$

The Suction Pump.—A diagrammatic representation of the suction or bucket pump is shown in Fig. 4·14 (*a*). The valves A and B are so constructed that they can only move upwards;



(*a*) A Suction Pump. (*b*) A Lift or Force Pump.

FIG. 4·14.

when the piston or bucket CD is forced downwards any water between the valves A and B is compelled to pass upwards through A, for the valve B is closed. When the motion of CD is reversed, i.e. the piston moves upwards, the water above it closes

the valve A and this water is carried upwards and delivered through the spout E. The space between CD and B would now be a vacuum were it not for the fact that the atmospheric pressure acting on the surface XY of the water in the reservoir forces the water past the valve B into the cylinder of the pump. On the descent of CD the cycle is repeated, the result being an intermittent delivery of water from the pump. In dry weather it is often necessary to *prime* such pumps, i.e. water must be poured into the main body of the pump in order to make an air-tight seal at CD. If such a process is not used the pump will not work.

The Lift Pump.—As in the preceding pump, there are two valves L and M, Fig. 4·14 (*b*), and an additional valve N is in a side exit. When the piston HK is raised the valve L closes, while M and N open, allowing water to pass from the reservoir into the cylinder below HK, while the water above HK is forced through N upwards into the cylinder. On the downstroke of the plunger HK the valves M and N close, and the water is forced through L into the receptacles which are being fed. The cycle of operations is then repeated. Vessels are often fitted to plunger pumps in order to provide a 'cushion' and so avoid damaging the pump when the piston motion is reversed. The air cushion absorbs the shocks which are due to the alternate starting and stopping of the water supply.

The Limitations of the Above Pumps.—Under normal conditions the pressure of the atmosphere is sufficient to support a column of mercury 30 in. in length. Since mercury has a density 13·6 times that of water the height of a water column which can be supported under similar conditions is $30 \times 13{\cdot}6$ in. or 34 ft. This distance represents the maximum theoretical distance between the water-level XY and the valve B, cf. Fig. 4·14 (*a*). In practice, owing to imperfections in the pump, it is seldom found that water can be raised more than 20 ft. by a suction pump.

This distance must not be confused with the height to which water can be driven by means of the force pump. This latter height depends upon the efficiency of the pump and the strength of the valves. A distance of 300 ft. is about the maximum distance through which it is safe to raise water in this way.

The Petrol Pump.—The lift pump finds a useful application in the modern petrol pump for raising petrol from an underground tank. When the plunger is raised by the ratchet work, R (shown in the conventional manner), Fig. 4·15, the valves V in the piston are closed and W is opened so that the petrol rises; on the descent of the plunger W automatically closes, thereby preventing the petrol from flowing back into the tank. At the same time the valves V are opened and the petrol is forced upwards into the glass vessel A, the air in A escaping through the outlet C. When A is filled, any excess of petrol

driven into it by the lift pump escapes down B and returns to the tank. The petrol in A is delivered through the tap T.

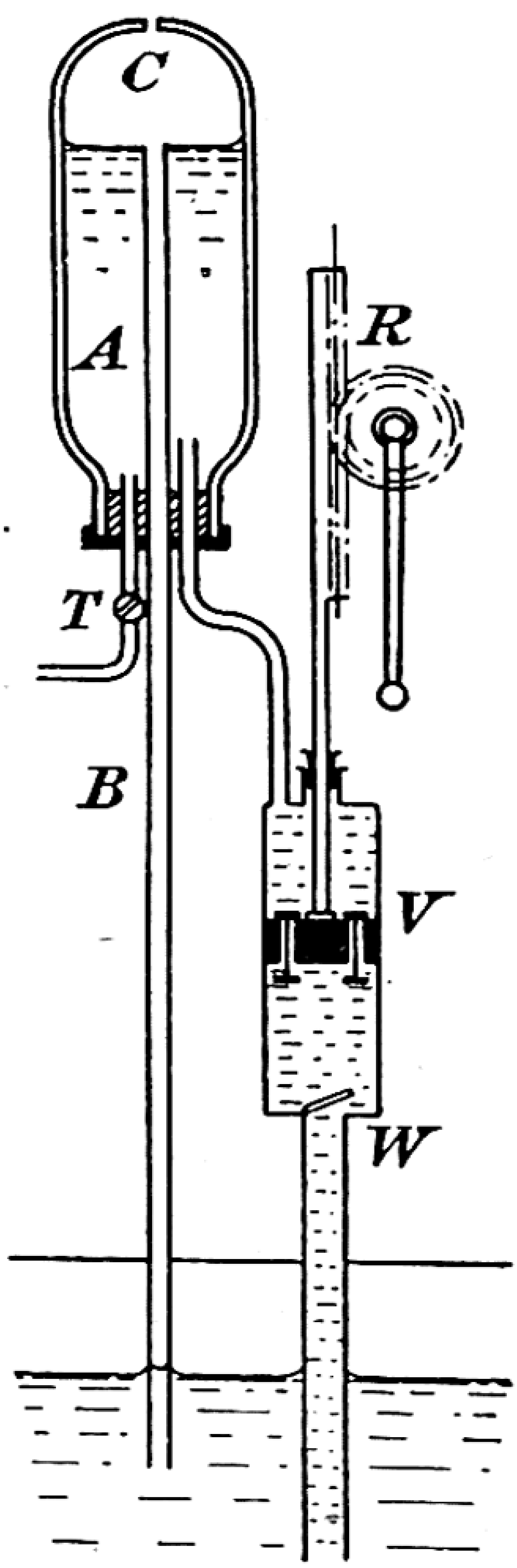
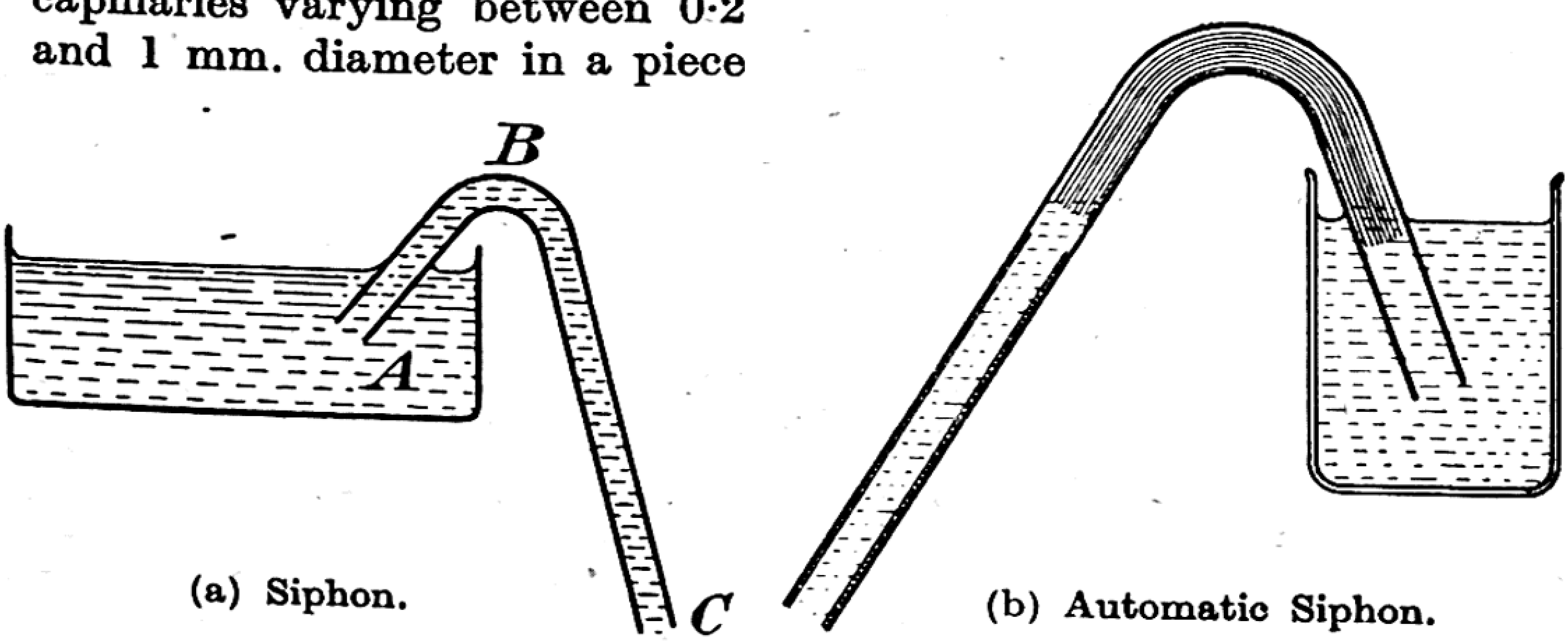


FIG. 4·15.—A Petrol Pump.

The Siphon.—The siphon, Fig. 4·16 (*a*), consists of a piece of tubing of rather small bore (0·5 cm.) bent so that its two arms are unequal. If the tube is filled completely with liquid and the shorter arm is immersed in a liquid, liquid is removed from the containing vessel. The column of liquid BC exerts a pressure at C, and when the siphon begins to operate the liquid runs out at C. The removal of the liquid from this side of the siphon tends to produce a vacuum in BA, and consequently the liquid is drawn from the reservoir, which is being emptied, into the tube. The whole process becomes continuous so that there is a steady stream of liquid at C. The speed at which the liquid is removed from its container depends upon the vertical distance between the level of the liquid and C; the greater the distance, the more rapid the flow from the siphon. However, it must be noted that if the vertical distance between A and B exceeds the barometric height, expressed in terms of the liquid in A, then the column AB can no longer be maintained and the siphon ceases to work. For water the above distance is 30 ft. (about); for mercury, 76 cm. The above argument indicates that a siphon will not work in a vacuum.

A siphon may be rendered automatic by placing some capillaries varying between 0·2 and 1 mm. diameter in a piece



(a) Siphon. (b) Automatic Siphon.

FIG. 4·16.

of straight glass tubing and bending the whole so that the shape shown in the Fig. 4·16 (*b*) is obtained. A little molten wax, made by melting together 10 parts resin and 6 parts vaseline, is drawn into the longer limb of the siphon so that the walls of the glass are thinly coated. When the shorter limb is placed in a liquid, capillary action causes some to pass into the waxed limb and form a pellet. This grows until the vertical distance between its ends exceeds the depth of the end of the short limb below the liquid surface. The ordinary action of a siphon ensues.

The Hydraulic Press.—A modern form of the hydraulic press first invented by BRAMAH is shown in Fig. 4·17. It consists essentially of a large cylinder, A, filled with water (or oil) in communi-

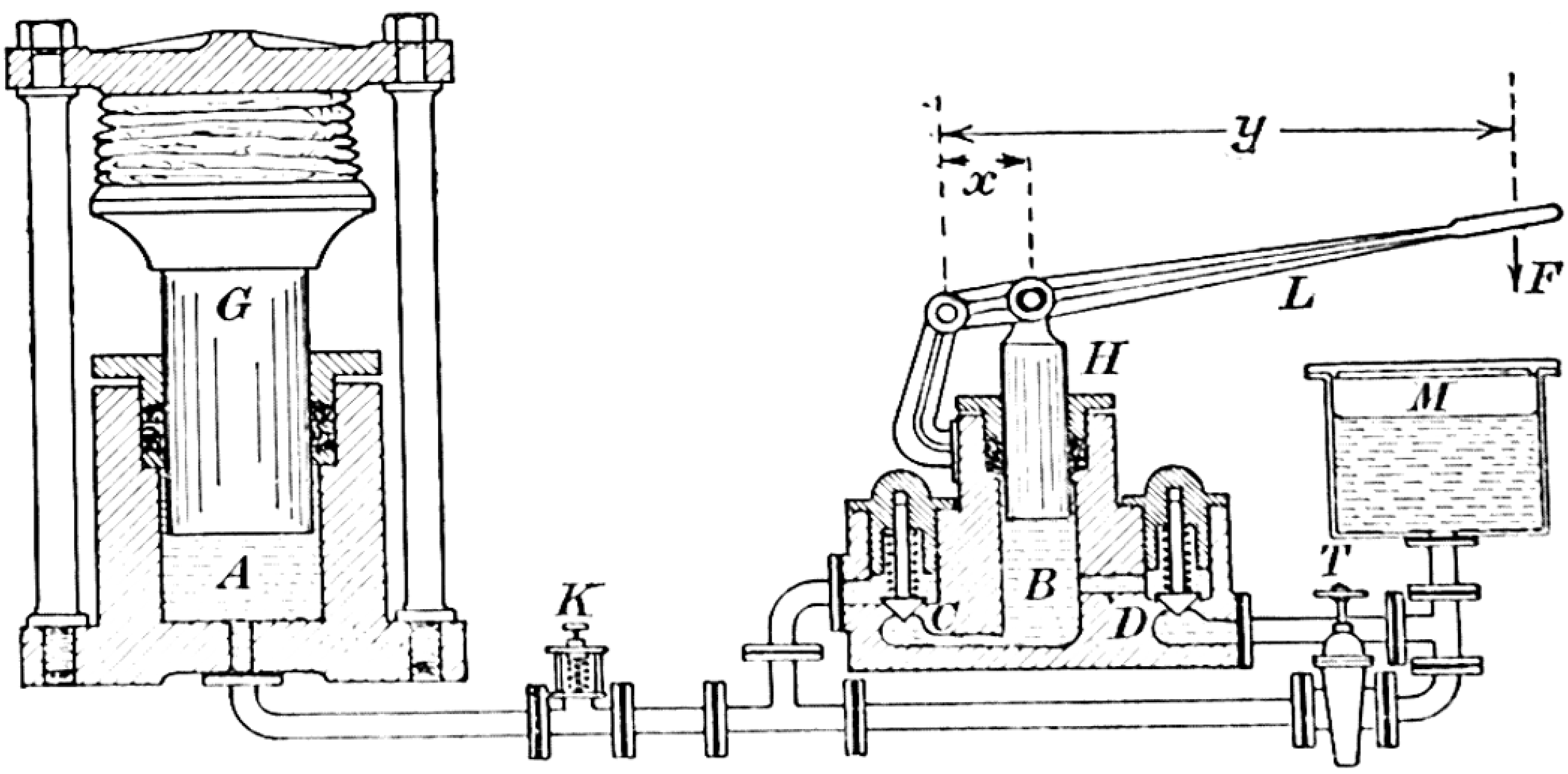


FIG. 4·17.—Hydraulic or Bramah Press.

cation with a smaller one, B. The larger cylinder is provided with a piston, G, known as the press-plunger, while the smaller one is provided with a piston, H, of much less cross-sectional area. It is termed the pump-plunger. Packing glands prevent the escape of liquid from the junctions between the pistons and the respective cylinders. H is operated by means of a lever, L, which further increases the mechanical advantage of the press. When a thrust is applied to the top of the smaller piston the pressure in B increases so that a valve, C, opens and the pressure is transmitted to the liquid in A. In consequence of this the press-plunger rises and compresses any goods carried on a platform attached to the top of G. When the lever is raised the valve C closes and D opens so that liquid enters B. The process may then be repeated. If, through some defect, the piston G fails to respond to the increased force acting upon it, the safety-valve K opens and the escaping liquid returns to the reservoir M via a channel not indicated in the diagram.

To release the pressure on the liquid in A the tap T is opened and the liquid returns to M.

If x and y are the perpendicular distances from the fulcrum of the lines of action of the thrust on the smaller piston and of the effort F applied to the extremity of the lever, the thrust on H is $F\left(\frac{y}{x}\right)$. If s is the area of cross-section of H, the pressure on the liquid in B is

$$\frac{F\left(\frac{y}{x}\right)}{s} = \frac{Fy}{xs}.$$

If S is the cross-sectional area of G, the thrust on its base is

$$F \cdot \frac{y}{x} \cdot \frac{S}{s}.$$

The mechanical advantage of this machine is $\frac{y}{x} \cdot \frac{S}{s}$, i.e. it is the product of the mechanical advantage of the lever and that of the simple press. [The machine is here considered to be an ideal one.]

It must be noticed that in the above argument we have assumed that the pressure on the base of H is exactly the same as that on the base of G. This is only true when these are in the same horizontal plane. If, at any instant, h is the difference in the above levels, the pressure difference is $g\rho h$, where g and ρ have their usual significance. The correction to be applied to obtain the pressure on the base of G is therefore variable; in general it is positive at the beginning of the stroke and negative at the end of it.

Air Pumps.—The simplest form of air pump is the glass filter pump shown in Fig. 4·18. The tube A is connected to the water supply, while the side tube C leads to the apparatus to be exhausted. A rapid stream of water is forced along A, and this produces a jet of water which passes down the tube B. The air in the immediate vicinity of B becomes entrapped in the water stream and is carried away through D. This process of entrapping the air is continuous until a pressure of about 3 cm. of mercury is reached—the pump then ceases to reduce the pressure further.

If a lower vacuum is required some other form of pump must be employed; if the space to be exhausted is not greater than 200 cm.^{3} the modified Toepler pump, Fig. 4·19, is very useful. It consists of a cylindrical barrel A, about 200 cm.^{3} capacity. At its upper end is a two-way capillary tap T; by turning this tap the barrel A can be put into connection, either with the tube B, which leads to the apparatus to be exhausted, or with C, which is open to the air. At the lower end of A is a smaller barrel D, with a side tap attached; any air entering the apparatus via

the pressure tubing is entrapped in D and can be removed through this side tap. D is connected to a mercury reservoir E, by means of pressure tubing.

To commence operations the reservoir E is raised, T being connected to C, so that the mercury fills the barrel A completely. T is closed ; E is then lowered a little and T rotated so that B and A are in connection. The pressure of the gas in B and the vessel to which it is attached forces the mercury downwards in A ; E is

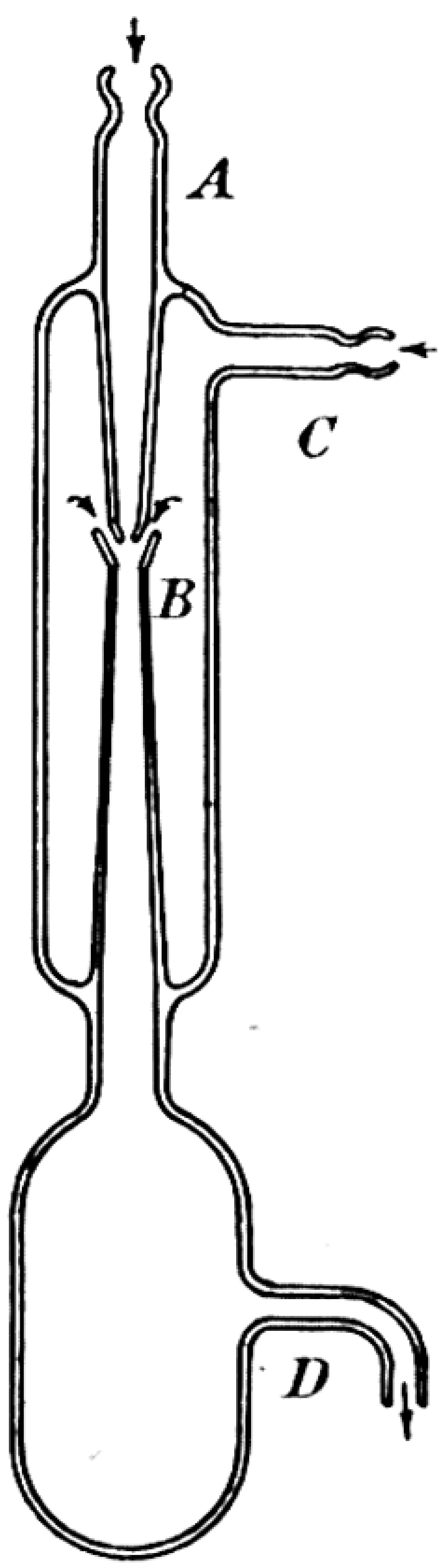


Fig. 4·18.—A Filter Pump.

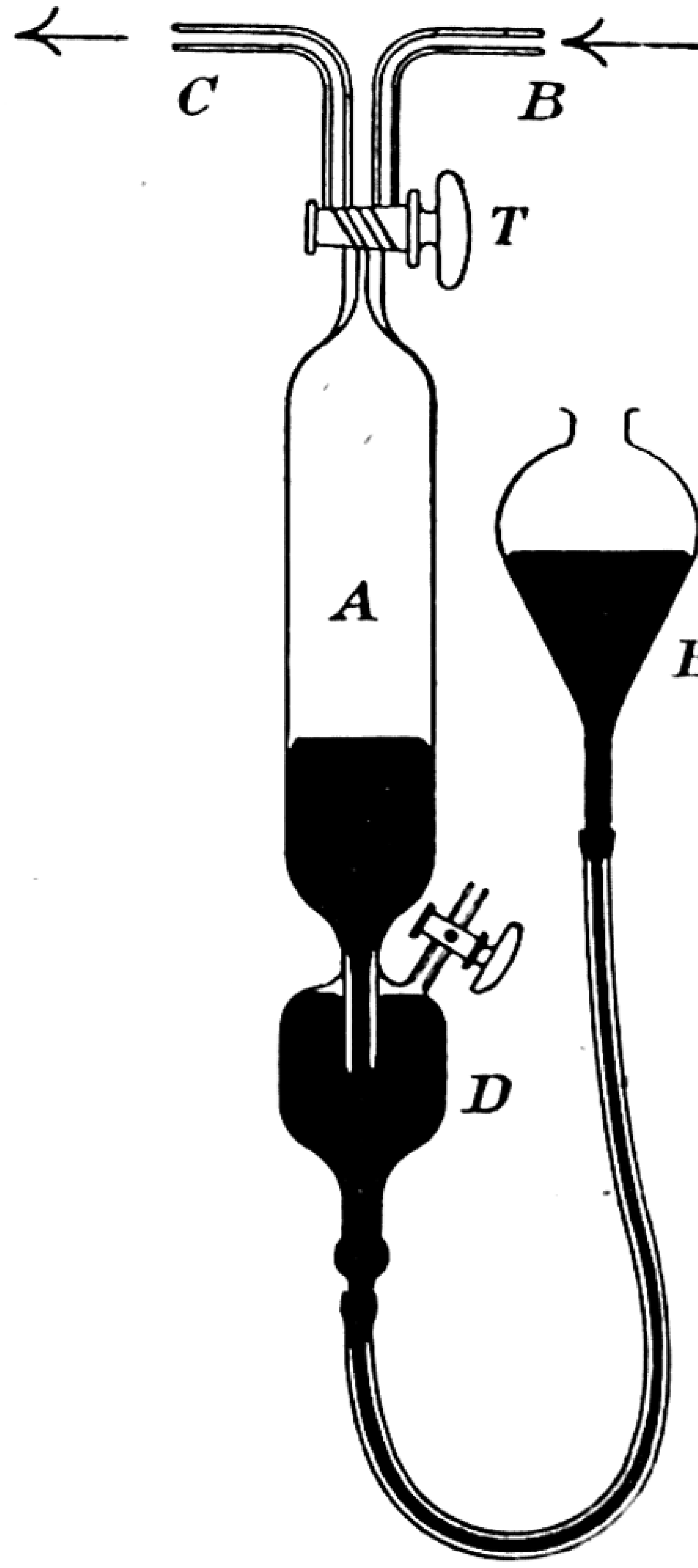


Fig. 4·19.—Toepler Vacuum Pump.

lowered until A is nearly filled with the gas. T is then closed and E raised until the pressure in A is greater than atmospheric. When this is so, T is put into connection with A and C so that the gas can be removed from A. The operation is repeated ten times or more, after which it will be found that no more gas can be removed from the vessel which is being exhausted. When the mercury in A reaches the tap T, the sound of a good metallic click indicates that a low vacuum has been reached.

The Sprengel Pump.—A form of this pump working in conjunction with a water pump is shown in Fig. 4·20. The capillary tubes in it are 0·15 cm. in diameter, the others about 0·5 cm. except where they widen out into bulbs approximately 2 cm. in diameter. The tube A leads to the vessel being exhausted. Pellets of mercury fall from the jet B and entrain bubbles of gas as they enter the fall tube below. The supply of mercury in B is replenished from the reservoir E which is in direct communication with a water pump. A capillary tube passes down the centre of this reservoir, through its base, and ends in the trough C. At the end of this tube there is a T-piece to which is attached a fine-drawn-out glass tube by means of a stout rubber tube. When the water pump is operating air is drawn in through this orifice and carries bubbles of mercury with it. When this mixture arrives at the upper end of the tube the air passes to the water pump while the mercury falls into the reservoir. A clip, K, controls the rate at which air enters the apparatus.

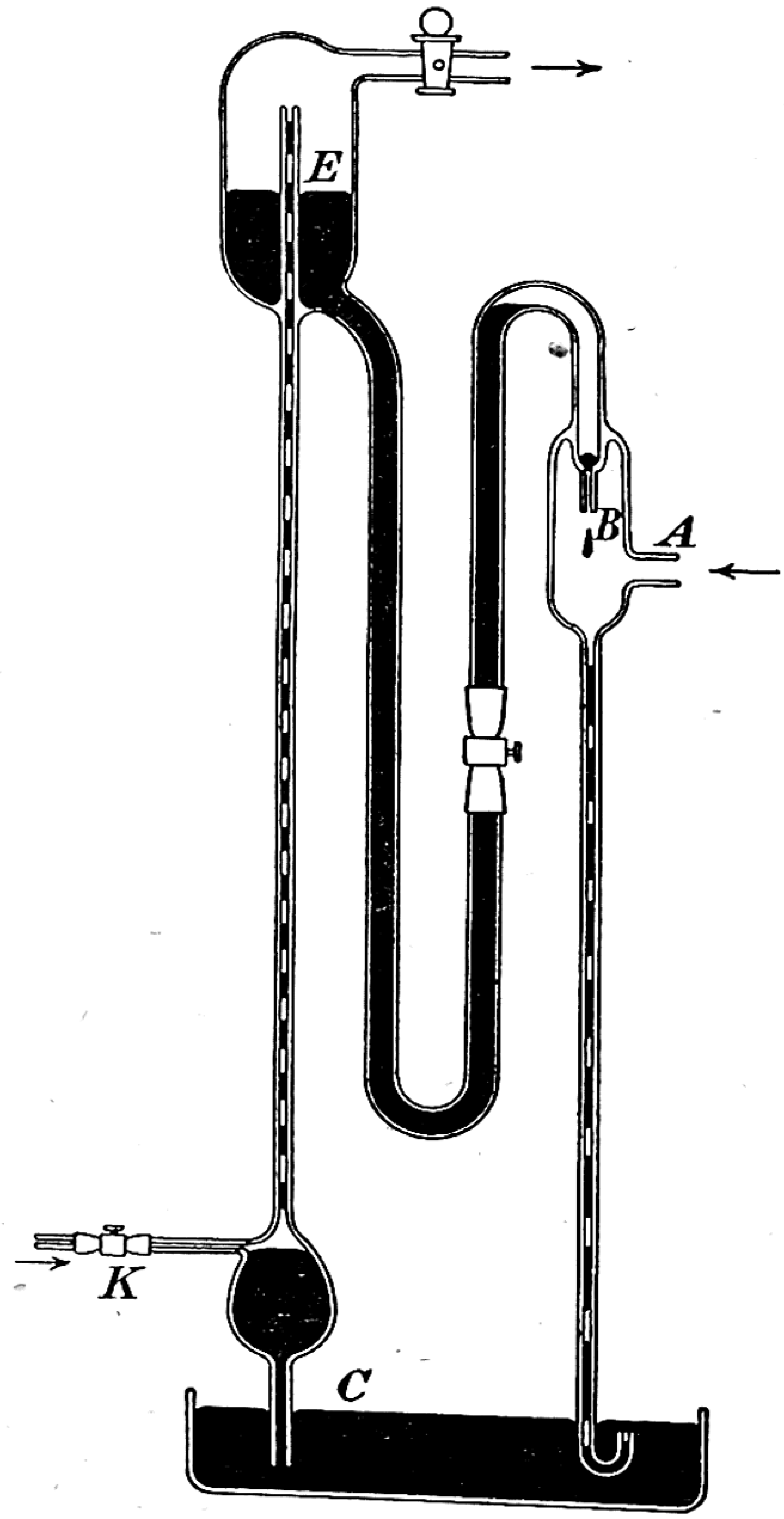


FIG. 4·20.—A Sprengel Pump.

High Vacua.—When the above procedure has been duly carried out, the degree of vacuum may be increased by having previously attached to the apparatus a bulb containing charcoal prepared from coconuts or cherry-stones. If the charcoal is reduced to the temperature of liquid air [$-$ 180° C.], it absorbs nearly all the residual gas and vapours [the Toepler pump will not remove vapours]. Instead of using charcoal, which is likely to explode at low temperatures if its gas content is high, it is better to use dried granular gelatinous silica in the bulb which is cooled, as this substance gives rise to no danger.

In the manufacture of wireless valves and X-ray tubes, mercury vapour pumps are employed to create a very high vacuum in them,

but these pumps can only be used with an auxiliary or 'backing' pump, i.e. the pressure in the apparatus must be low [$<$ 1 cm. of mercury] before they will work. The mercury vapour pump described below is capable of producing an X-ray vacuum when backed by a filter pump, but the best mercury vapour pumps require to be backed by a rotary vacuum pump—cf. the next section. The modern condensation pump was originally designed by LANGMUIR, but nowadays there are many patterns. One designed by WARAN is shown in Fig. 4·21. Mercury is boiled in a vessel A [since the pressure is low, the temperature is seldom above 180° C.] and a mercury vapour jet is formed at C.

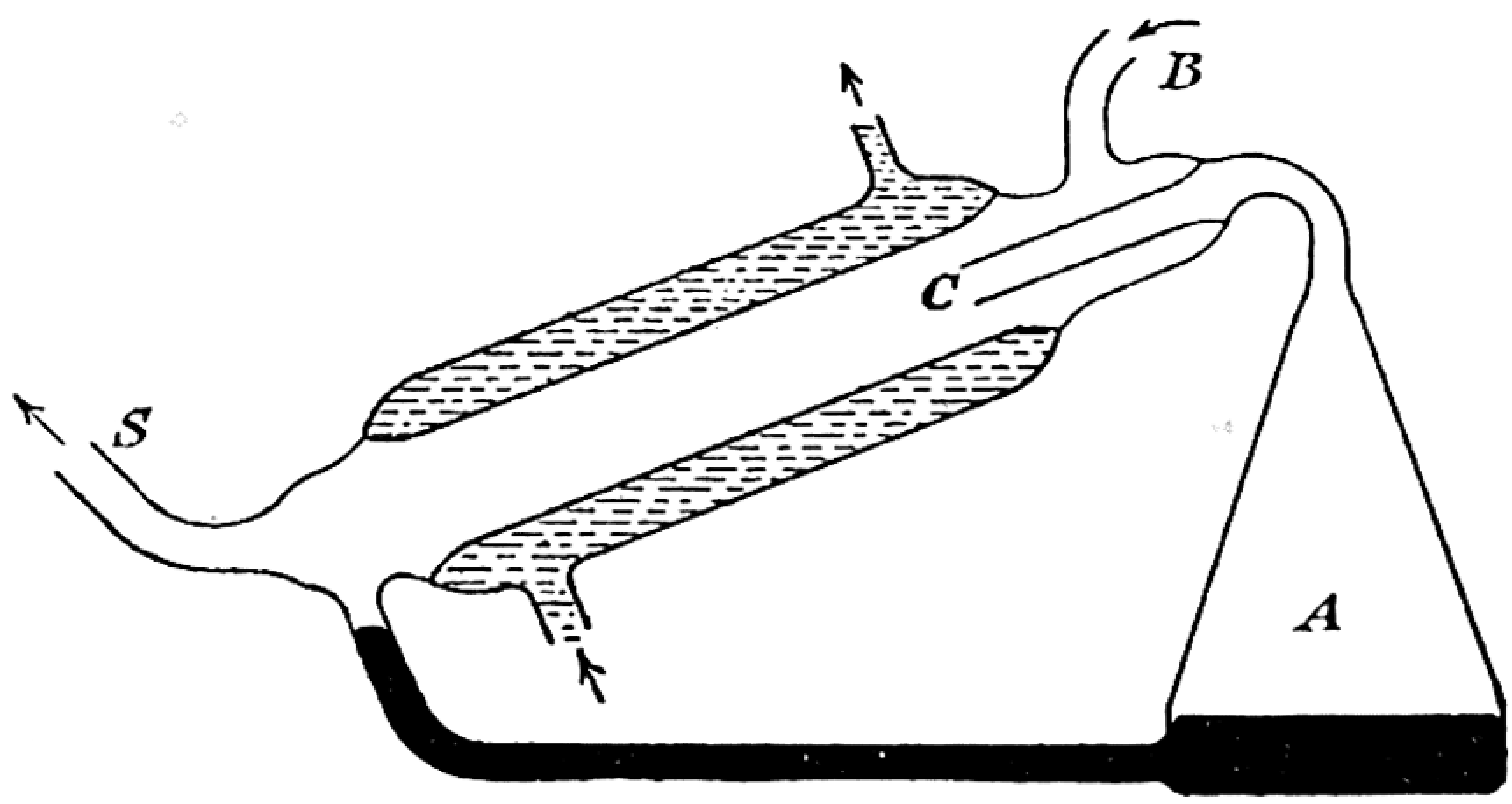


FIG. 4·21.—Mercury Vapour Pump (or Diffusion Pump).

The vessel to be exhausted is connected at B, whilst a water pump is attached to S. Around the wide tube into which the nozzle C projects, there is a water jacket, through which a constant stream of water flows. Consider the state of things in the neighbourhood of the jet. Molecules of mercury vapour and of the gases will tend to intermingle. They are said to diffuse. The mercury vapour, which diffuses towards B, is condensed, whereas the gaseous molecules diffuse towards S and are withdrawn by the water pump. In this way a very low vacuum is reached, but one must not imagine that *all* the molecules have been removed even in the highest vacua which have been produced. There still exist in such vacua about twenty millions of molecules per mm.^{3}.

A Rotary Vacuum Pump.—The pump shown in Fig. 4·22 is designed for the production of a high vacuum and the exhaustion of vessels of large capacity. It works directly from atmospheric pressure and being entirely immersed in oil the leakage of air into the high vacuum is prevented. The pump consists of an outer steel casing, C, through which is bored a cylindrical chamber, D. A shaft, M, runs through this chamber, its axis being parallel to but eccentric from the axis of the chamber. This shaft revolves about its own axis and always touches the periphery of the chamber D at the point E. On each side of this point is a port—one an inlet, F, and the other an outlet, G, which is fitted with a spring-loaded valve, H. In the shaft M is a slot in which two plates, P and Q, are free to slide to and from the axis of the shaft. These two plates are kept apart and their extreme edges forced against the periphery of the chamber D by a series of springs placed at right angles to the axis of the shaft—one of these is shown in sectional view.

The action of the pump is as follows. Let us consider the position shown in the diagram. The shaft M is rotating in an anti-clockwise direction and the effective space between the chamber D and the shaft M is divided into two portions, S and T. As the shaft rotates, remem-

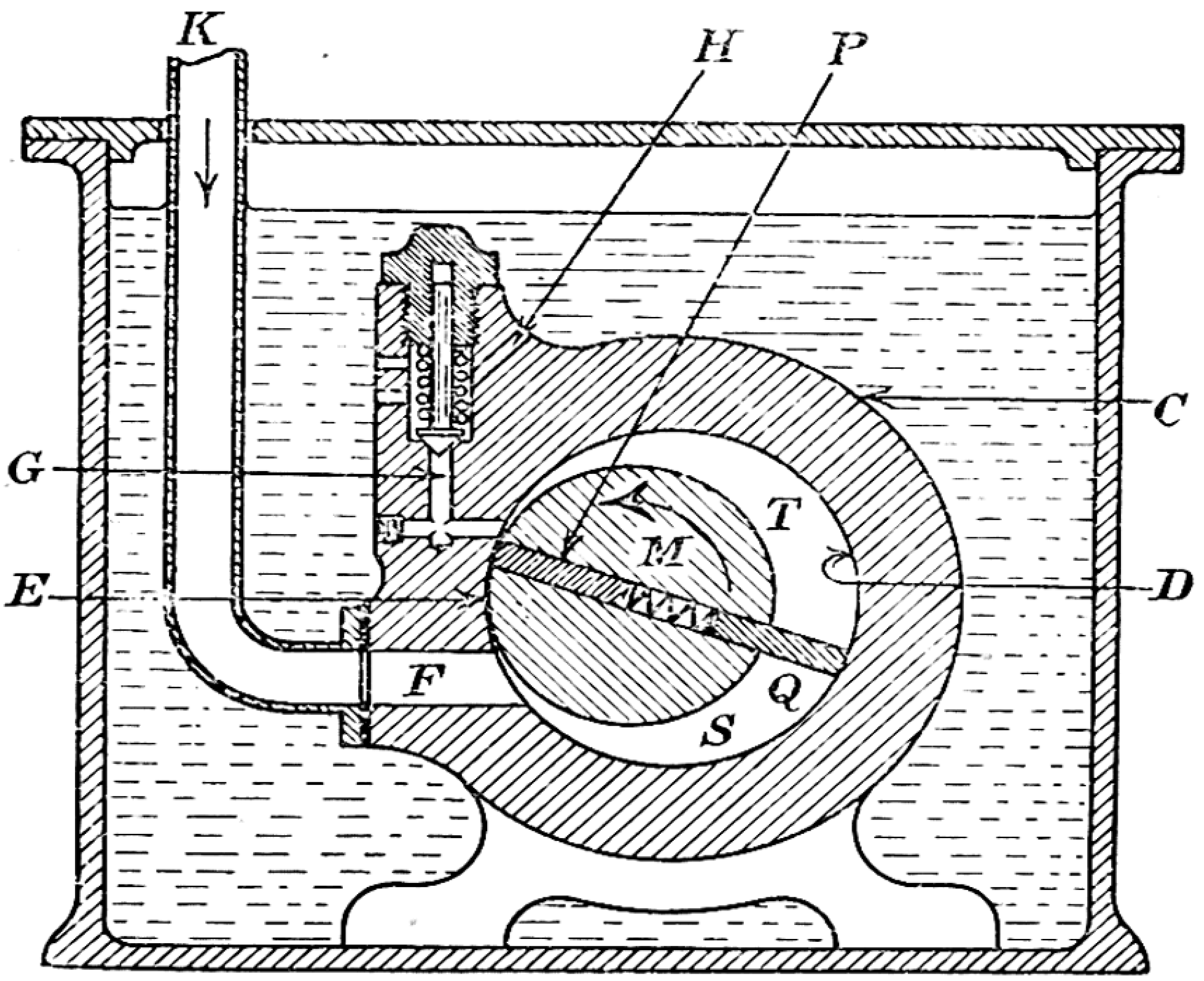


FIG. 4·22.—A Rotary Vacuum Pump.

bering that the plate Q is touching the wall of the chamber, the portion S enlarges and air is drawn in from the vessel to be exhausted through the inlet pipe K. The portion T is getting smaller and any air in it will be compressed. When the pressure is sufficiently great this air escapes through the exhaust valve. Thus the pump will exhaust air from a vessel to which the inlet pipe K is connected.

The Measurement of Low Pressures.—When it is necessary to know the pressure inside a partially exhausted vessel a manometer is used. This consists essentially of a U-tube closed at one end. The closed end is ***completely*** filled with mercury but there is only a small amount in the other limb of the tube. When the manometer is connected to a vessel from which the gases are being removed gradually, a point is finally reached when the mercury begins to descend in the closed limb of the tube. Finally the difference in level between the mercury surfaces in the two tubes becomes constant and is then a measure of the pressure of the remaining gas in the vessel which is under evacuation. Such manometers possess several disadvantages :—

(*a*) The vacuum in the closed limb is gradually destroyed by gases which creep between the mercury and glass surfaces.

(*b*) If the apparatus suddenly develops a leak the mercury is forced rapidly into the closed limb and the impact is sufficient to cause a fracture of the manometer.

(*c*) The instrument is not sensitive at low pressures.

(*d*) The mercury tends to stick to the glass so that it becomes difficult to observe the true pressure.

The first two disadvantages can be minimized by the use of a device due to WARAN. A small glass reservoir R, Fig. 4·23, is joined by means of capillary tubing to the usual form of manometer. The whole is filled with mercury as before. When the pressure upon the free surface of the mercury is diminished, at some stage the mercury recedes from the point A. If at this stage the instrument is tapped gently, the continuous thread of mercury in the capillary tube is broken and the mercury assumes the position shown in the diagram. The capillary tube space is then an almost perfect void, so that the height BC is a true representation of the pressure at C.

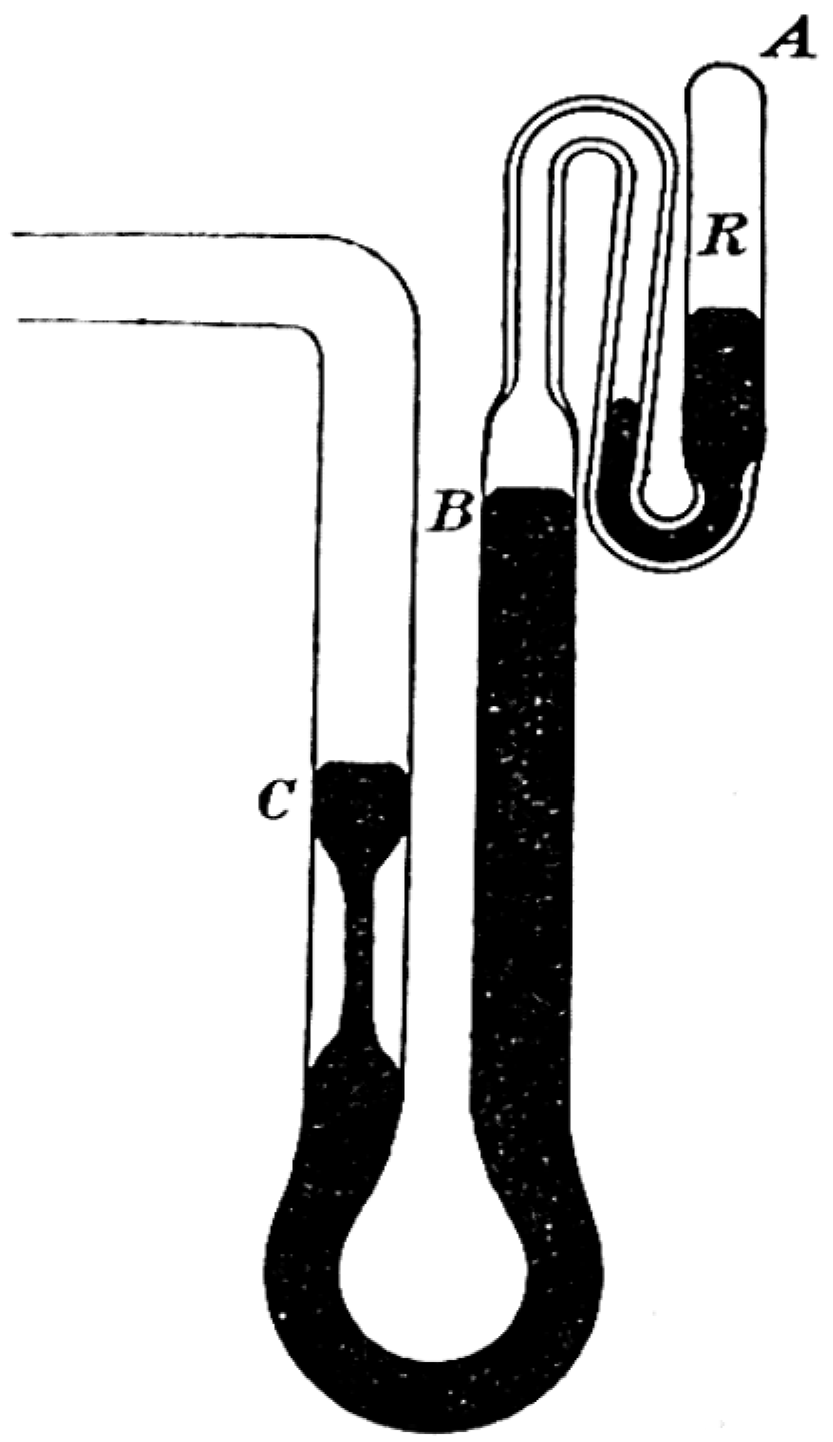


FIG. 4·23.—Manometer with Regenerative Vacuum Device.

After some time gases may make their appearance in the capillary; they are removed by subjecting the manometer to atmospheric pressure, thereby forcing them into R. By constricting the open limb of the U-tube as shown in the diagram, the motion of the mercury is retarded so that a fracture from the causes mentioned above becomes a very remote possibility.

The McLeod Gauge.—Since it is impossible to use a mercury manometer to measure high vacua (such as exist in wireless valves) it is important to discover a means whereby this may be done. McLEOD is responsible for the gauge which is frequently used for this purpose. A bulb A, Fig. 4·24 (*a*), of known volume V, has fixed to its upper extremity a capillary tube DE, the volume of which per unit length is known. The tube BC leads to the apparatus in which it is desired to measure the pressure. A reservoir F contains mercury and is attached to the gauge proper by means of pressure tubing G. When the reservoir F is lowered through a distance greater than that equal to the barometric height (say 80 cm.) below the level B, then A is in direct contact with the exhausted vessel, and is therefore filled with gas at a pressure p, which is the pressure to be determined. When F is raised, the mercury divides at B and entraps a volume V of gas at pressure p; by raising F still more this gas can be compressed into the capillary DE. To derive a value for p the mercury in C may be adjusted until it is level with the closed end D of the capillary tube. Then the pressure of the gas in DE is measured by h, where h = DE. Now Boyle's law [cf. p. 85] states that the product of the pressure (p) and the volume (V) is constant for any given mass of gas at constant temperature. Applying this to the mass of gas entrapped in the capillary, we have

$$pV = hx,$$

where x is the volume corresponding to the length DE of the capillary tube. Whence

$$p = \frac{hx}{V} = \frac{h^2v}{V},$$

if v is the volume per unit length of the capillary.

If such a gauge is to be reliable the enclosed gas must be dry, for water vapour does not behave like an ideal gas.

In the more recent forms of this instrument a piece of glass tubing of the same diameter as that used for DE is sealed in parallel with the side tube C as shown in Fig. 4·24 (*b*). When reading the difference in levels of the mercury in the tube E and that leading to the vacuum, it is the levels in E and this other tube which must be recorded. This is because the surface tension of mercury is such that it is depressed in narrow tubes to an extent depending on the diameter of the tube. The effect is eliminated, however, by using tubes of the same diameter.

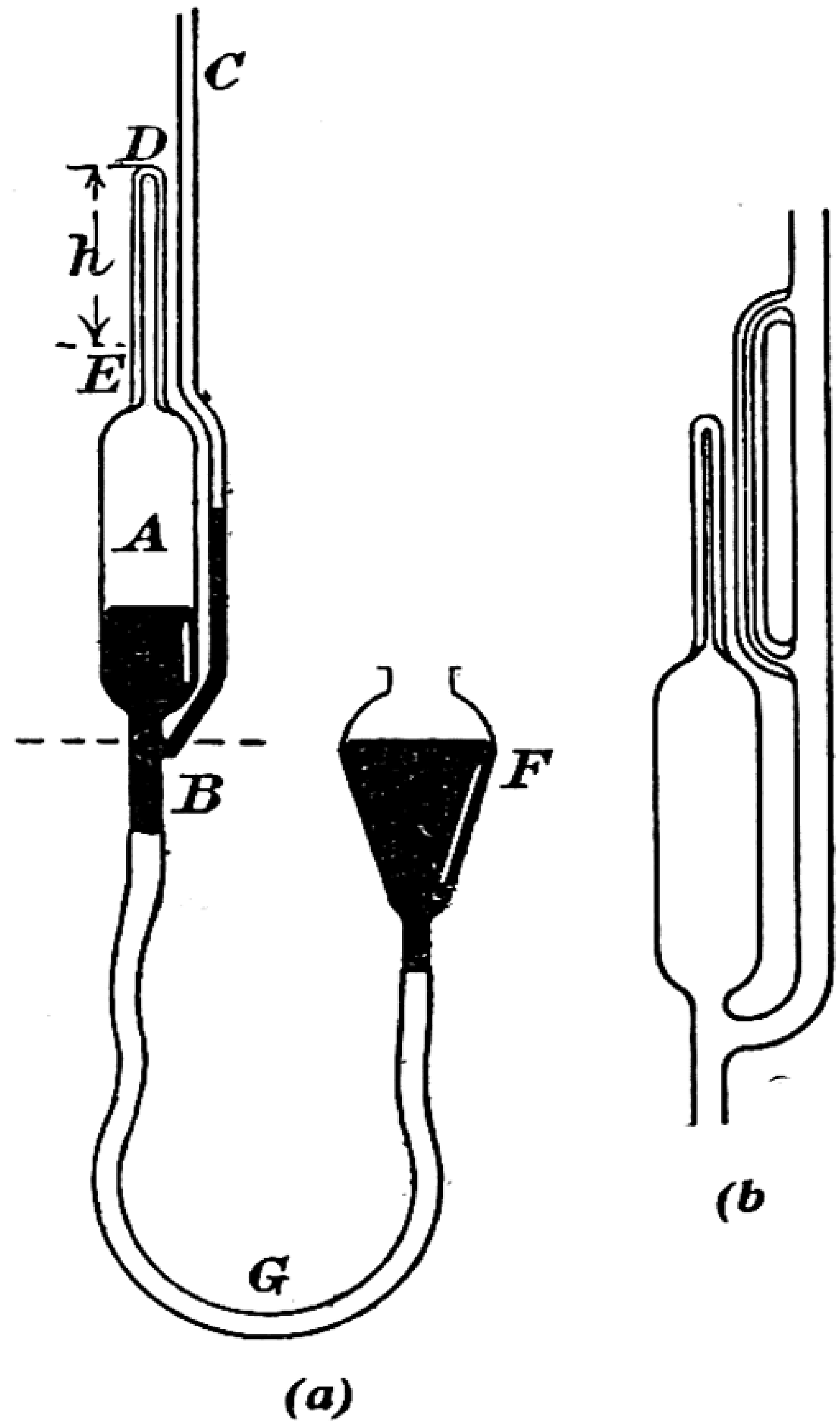


FIG. 4·24.—McLeod Gauge for Measuring Low Pressures.

The Absorption of Gases.—The process of obtaining a high vacuum is by no means as simple as the above remarks would indicate. It is found that after a certain time, depending on the pump and the nature and size of the vessel to be exhausted, the pressure ceases to be reduced. This is because gases are evolved from the surfaces of all substances when the external pressure is very low. The rate at which these gases is expelled is greatly increased when the temperature of the surface is raised. The vessels to be exhausted are therefore heated cautiously with a gas flame and the pumping continued.

If, as in a wireless valve, there is some metal to be degassed, it is subjected to a heavy electron bombardment. We shall learn later that electrons are emitted when a metal is heated to high temperatures. A filament is therefore placed near the metal (or the filament of the valve used) and its temperature raised electrically. A large positive potential is then applied to the metal, while the filament is earthed at one point. The electrons are attracted to the metal and strike it with considerable velocity. They lose their kinetic energy which appears as thermal energy [heat], and it is this energy which is responsible for the liberation of the occluded gases in the metal.

EXAMPLES IV

1.—Calculate the mass of lead, density 11·3 gm. cm.$^{-3}$, which must be attached to 105 cm.^{3} of wax (density 0·86 gm. cm.$^{-3}$) in order that the apparent mass may be zero when the whole is placed in a liquid whose density is 1·04 gm. cm.$^{-3}$.

2.—A U-tube contains mercury, density 13·6 gm. cm.$^{-3}$. A liquid whose density is 1·23 gm. cm.$^{-3}$ is poured into one limb so that the difference between the mercury levels is now 3·67 cm. What is the length of the column of liquid? Can you make any statement concerning the mass of the liquid which has been added?

3.—The height of a water barometer is 34 ft. Find the pressure in atmospheres 1 mile below the surface of sea water (density 1·026 gm. cm.$^{-3}$). Also express this pressure in ton.-wt. ft.$^{-2}$ [1 cu. ft. of water has a mass of 1000 oz.].

4.—Find the pressure due to a column of air 1 mile high if the density of the air is uniform and equal to 0·00129 gm. cm.$^{-3}$. Describe how a barometer may be used to determine the height of a mountain.

5.—A rectangular tank measures 4 ft. by 3 ft. at the base. It is filled with water to a depth of 8 in. What is the depth when a stone (1 ft. cube) is dropped into the tank?

6.—What do you understand by the principle of flotation? An iron cylinder 12·0 in. long floats vertically in mercury. The densities of iron and mercury are 7·8 and 13·6 gm. cm.$^{-3}$ respectively. Calculate the length of iron immersed.

7.—Define the term density. How would you proceed to determine the density of a powder such as plaster of Paris?

8.—How would you determine the density of a newly-laid egg?

9.—Sketch and describe the experimental arrangement you would use in order to obtain a good vacuum. How would you measure the final pressure obtained?

10.—A piece of glass tubing sealed at both ends has a mass 18·26 gm. If the density of glass is 2·63 gm. cm.$^{-3}$, calculate the volume of the air space enclosed in the bulb if the whole has an apparent mass of 6·37 gm. in water.

11.—The space above a mercury column contains some air. The mercury column is 28·40 in. long and the space above is 3·05 in. long. This tube is then pushed downwards into mercury so that the column is 28·14 in. whilst the air space is 2·34 in. What is the true height of the barometer?

12.—What mass of lead, density 11·3 gm. cm.$^{-3}$ must be added to a block of Balsa wood 3·26 cm. $\times$ 8·40 cm. $\times$ 9·62 cm., and density 6·0 lb. per cu. ft., so that it will just float in water? [1 lb. = 453·6 gm., 1 ft. = 30·48 cm.]

13.—A pellet of mercury, density 13·59 gm. cm.$^{-3}$ mass 5·278 gm., has a length 20·4 cm. when introduced into a narrow tube. What is the average radius of this tube? Some liquid is then placed inside the tube and the length of the column is 18·9 cm. What is the density of the liquid if its mass is 0·467 gm.?

14.—What is meant by the statement that the pressure of a coal gas supply is 12 cm. of water? If the pressure of the gas supply at ground-level is 12 cm. of water what will be the pressure of the supply at the top of a building 25 metres high if the relative densities of gas, air, and water are as 1 : 2 : 1,450?

15.—Explain the conditions on which floating depends. A cork of

specific gravity 0·25 floats in sea-water of specific gravity 1·25 with 10 cm.^{3} above the surface. Calculate the total volume of the cork.

16.—Define density. If the density of glass is 2·265 gm. cm.$^{-3}$, express its density in terms of the lb. and yard when these are the units of mass and of length respectively. [1 lb. = 453·6 gm., 1 in. = 2·540 cm.]

17.—If you were supplied with some turpentine and some ice, describe how you would determine the density of the ice without using any form of balance or 'weights.'

18.—A body 'weighs' 86·0 gm. in air, 72·4 gm. in one liquid and 63·9 gm. in another liquid. In a mixture of these liquids it 'weighs' 67·1 gm. Calculate the proportion in which the liquids have been mixed.

19.—A solid whose density is 12·4 gm. cm.$^{-3}$ is weighed in air. It is found that its mass is 284 gm. when brass weights having a density 7·8 gm. cm.$^{-3}$ are used. If the density of air is 1·25 gm. litre^{-1}, calculate the error due to neglecting the buoyancy of the air.

20.—A cylinder of 0·3 cm.^{2} cross-section is loaded at one end and the whole has a mass of 6·43 gm. In water it is found that 1·8 cm. project above the surface. Calculate the amount of this projection when the cylinder floats upright in a liquid whose density is 1·37 gm. cm.$^{-3}$.

21.—Describe a modern form of barometer. What is a bar? Calculate the number of bars in one standard atmosphere.

22.—The pressure at a depth of 100 ft. in a fresh-water lake is three times the pressure at a depth of 11 ft. Determine the height of the mercury barometer in cm. [Density of mercury = 13·6 gm. cm.$^{-3}$.]

23.—A column of mercury is placed at the middle of a uniform glass tube and both ends of the tube are closed when the tube is horizontal, and the pressure everywhere 76 cm. of mercury. The tube is then placed vertically and it is found that the length of the tube occupied by the air above the mercury is twice as great as that occupied by air below the mercury. What is the length of the mercury column?

24.—If a series of observations of the volume, V, of dry gas enclosed in a Boyle's law apparatus and the excess pressure (p) inside the apparatus were made, explain how the atmospheric pressure may be deduced from a graph showing the relation between p and $\frac{1}{V}$.

25.—Describe how you would proceed to verify Boyle's law. The height of a faulty barometer which has a little air in the space at the top of the mercury column is 28·6 in. when the barometric height is 29·1 in., and 29·2 in. when the true height is 30·1 in. Calculate the barometric pressure when the instrument indicates 28·9 in.

26.—The following figures are taken from the treatise in which Boyle published an account of one of his experiments made to determine the relation between the pressure and volume of *a given parcel of air* at room temperature. Use them to find a value for the height of the barometer on the day when this experiment was made.

Length of tube occupied by air (inches)	$11\frac{1}{2}$	$10\frac{1}{2}$	9	8	$5\frac{3}{4}$	$4\frac{1}{2}$	3
Excess pressure of the air inside the tube over atmospheric pressure outside (inches of mercury)	$1\frac{7}{16}$	$4\frac{6}{16}$	$10\frac{2}{16}$	$15\frac{1}{16}$	$32\frac{3}{16}$	$48\frac{12}{16}$	$88\frac{7}{16}$

CHAPTER V

CONCERNING THE NATURE OF FLUIDS

The Brownian Movement.—To an observer standing on the landward side of a breakwater the nature of the tempestuous seas beyond that breakwater can be inferred from the rolling and pitching motions of the ships which will be more excessive than usual. To the eye, aided by the most powerful of microscopes, the motion of molecules cannot be made visible. If, however, some small particles of gamboge suspended in a liquid are observed with the aid of a microscope, it will be found that these particles are always moving, not in any fixed direction, but in all random directions. The actual motion of a particular particle is very irregular, and perhaps the most striking feature of this phenomenon is that the motion never ceases. This phenomenon, discovered by an English botanist BROWN early in the last century, has been observed in liquids contained in the enclosed cavities of some varieties of quartz, and these cavities and the liquids in them will have been there for thousands of years. It has been concluded that this eternal motion of the suspended particles cannot be due to any external agencies, but must be attributed to the movements of the molecules which constitute the liquid.

The Brownian motion can also be detected in ***collosol oil of iodine***. This substance is applied to the patient's skin in cases where it is necessary to alleviate the pain due to rheumatism, sciatica, etc. The small particles of iodine are participating in this so-called Brownian movement, and consequently they are able to pass very readily through the skin and into the body.

Diffusion.—Let a quantity (say 25 cm.^{3}) of a concentrated nickel (or copper) sulphate solution be placed at the bottom of a tall glass cylinder, the remainder of the vessel being filled with water. A glass cover prevents evaporation. Such a coloured substance is chosen so that the movements of the resulting solution may be observed easily. At first the line of demarcation between the water and the solution is well defined, but it becomes obliterated after a lapse of several days. The dissolved substance has moved upwards against the pull due to gravity, i.e. it has moved to a region where

the concentration of the salt in solution was less. The rate at which this transference of the dissolved substance takes place is very slow. It would be very difficult to explain this phenomenon if the molecules of the liquids were not in a state of continual irregular motion. The molecules of the dissolved substance—or, in the case of electrolytes, the ions in the solution—behave, in this respect, like the molecules of a gas, and the process by which molecules in different solutions move from regions of higher to those of lower concentration, or the molecules of one gas intermingle with those of another is called ***diffusion.*** In a gas the molecules are at relatively large distances from one another and so are free to move. The molecules of the dissolved body in a solution may be regarded, for some purposes, as being distributed throughout the solvent ; the solvent has merely made it possible for the constituent molecules of the dissolved body to occupy a space much beyond the original confines of the crystal.

The Diffusion of Salts in Aqueous Solution.—In 1850 GRAHAM published his first paper on the diffusion of salts in solution, and in 1882 a further study was made by SCHEFFER. In principle the apparatus they used is shown in Fig. 5·1 (*a*). A small glass cylinder, A, rests on two horizontal glass rods supported inside a larger glass vessel, B. A is nearly filled with the solution under investigation, and a cork, C, floats centrally on the liquid. A vertical knitting needle attached to this cork can move upwards in a narrow glass tube, D, held in position by a clamp and stand (not shown). By this means the cork is kept in a central position. Water is contained in the dropping funnel, E, and it is allowed to drop on to the top of the cork, which has been thoroughly wetted, at the rate of about three drops per second. A layer of water soon appears on top of the solution, and when the cork is clear of the solution, it may be removed, and the vessel, A, completely filled with water. The whole of A is then surrounded by water as in Fig. 5·1 (*b*). The temperature is kept constant to avoid convection currents. At first there is a distinct boundary between the solution and the water. As a result of the process called

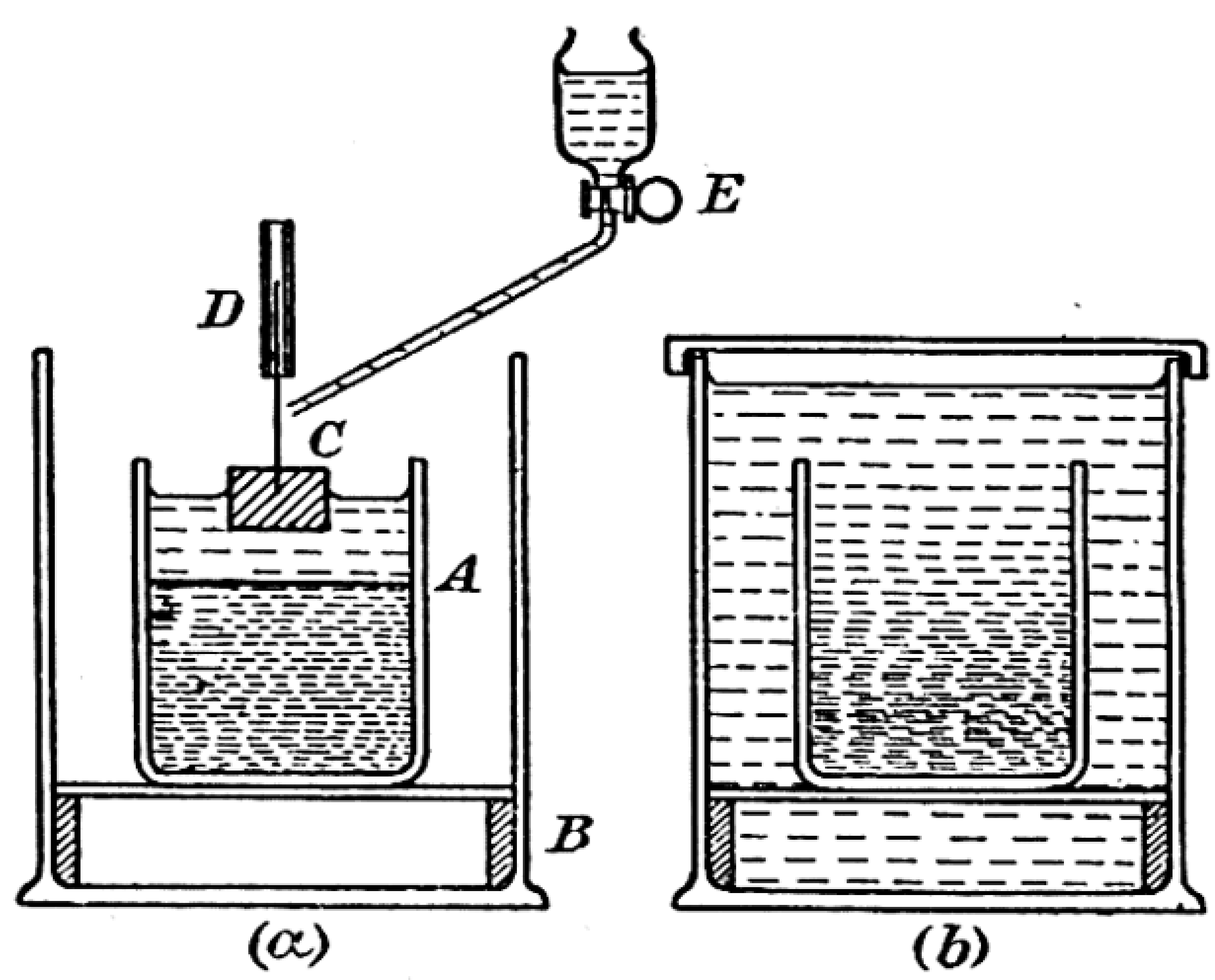


FIG. 5·1.—The Diffusion of Salts in Aqueous Solution.

diffusion this well-defined boundary soon disappears. By determining the amount of solute which had escaped from the inner vessel into the outer one, it was found

(i) the rate of diffusion depends on the nature of the dissolved substance, so that the ratio of the amounts of two substances present in a solution may alter on account of diffusion,

(ii) the rate of diffusion is directly proportional to the concentration of the dissolved substance,

(iii) a rise in temperature augments the rate at which diffusion takes place.

Fick's Law.—Four years after the publication of Graham's first paper on diffusion, Fick, guided by Fourier's work on the conduction of heat, enunciated the following law. ***The mass, m, of a substance in solution passing across an area A per second is directly proportional to the rate at which the concentration, c, of the dissolved substance diminishes in a direction at right angles to the plane of the area A.*** In symbols

$$\frac{m}{A} = -D\frac{dc}{dx},$$

where D is the coefficient of diffusion, and $\frac{dc}{dx}$ is the rate at which the concentration increases with the distance x.

The Passage of Gases through Porous Bodies.—The diffusion of two gases is not prevented but only hindered when a thin porous wall or membrane separates them, but the actual rate at which the gases intermingle depends upon several factors. If the pores through which the gas passes are short in comparison with their diameters the gas flow is similar to that of water through a hole in the side of a thin-walled container. This process is known as ***effusion.*** The velocity of effusion is proportional to $\sqrt{\frac{p}{\rho}}$, where p and ρ are the excess pressure of the gas above that of the surrounding air and the density of the gas or gas mixture passing through respectively. When the pores are reduced in diameter the flow of gas, for a given difference in pressure between the ends of the tube or pore and provided that the pressure difference is not so large that turbulent motion ensues, is controlled by the viscosity of the gas [cf. p. 130]. In both these instances the gas passes through as a whole so that if it were a mixture of gases no partial separation would be effected. Conditions are very different, however, when the pores are so fine that their diameters are comparable with those of the gas molecules. Graham, who first investigated these phenomena about 1840, discovered that the rate of diffusion at a given temperature was directly proportional to the difference

in pressure between the two sides of the membrane, and inversely proportional to the square root of the density of the gas. This is known as ***Graham's Law of diffusion for gases.***

Hence, for a given pressure difference, hydrogen diffuses four times as quickly as oxygen through the same membrane, since, under the same conditions, the density of a gas is directly proportional to its molecular weight. This implies that if an oxygen-hydrogen mixture is introduced under pressure into a porous vessel the mixture passing through will be four times as rich in hydrogen as in oxygen.

The diffusion of gases through porous media may be investigated

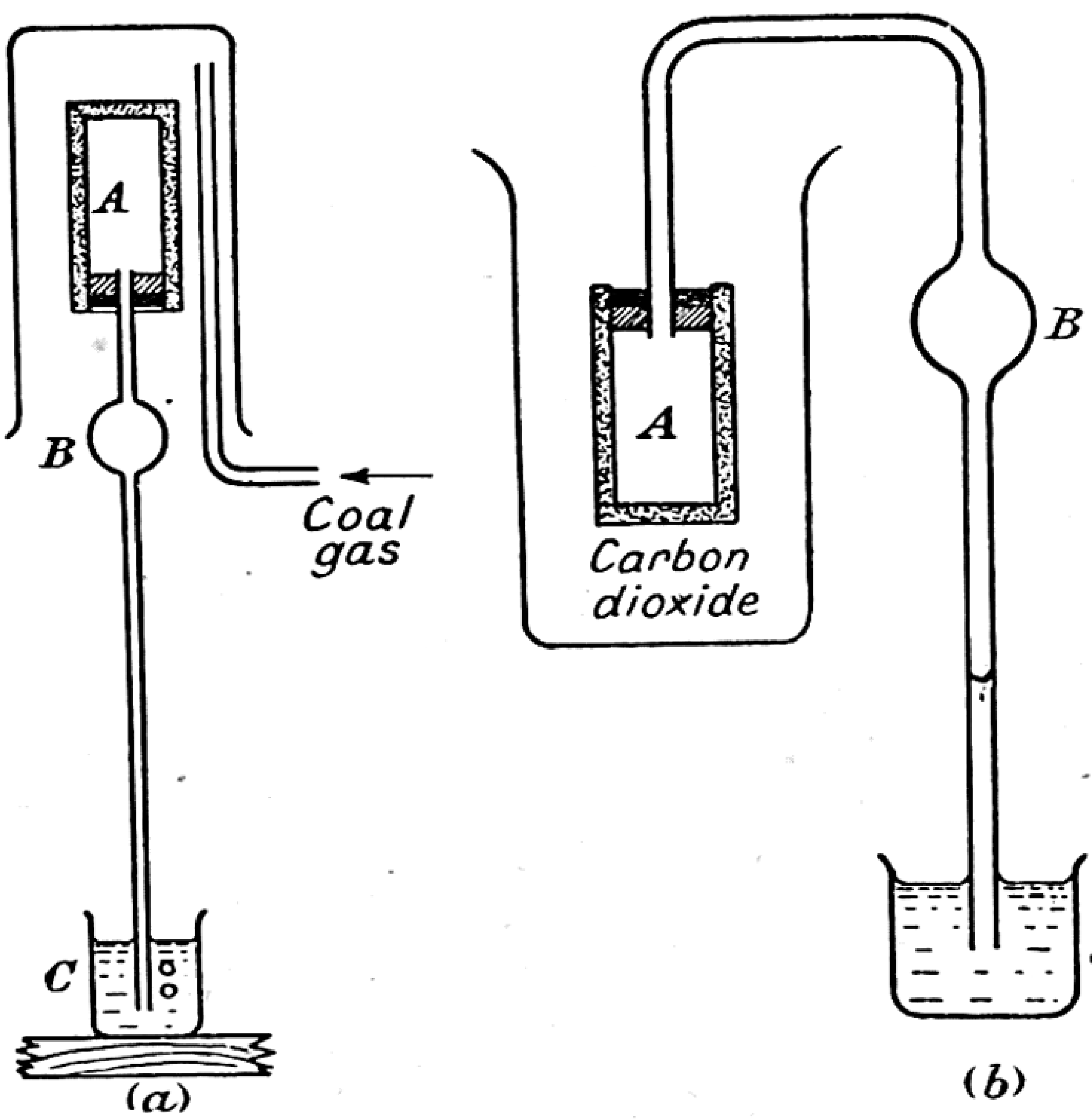


FIG. 5·2.

experimentally with the aid of the apparatus shown in Fig. 5·2 (*a*). A glass tube 60 cm. long and 0·5 cm. wide passes through a cork from a porous pot A to a vessel containing coloured water. The cork is pushed well within the pot and covered with sealing-wax to make the joint air-tight. A large jar is held over the pot and coal gas introduced into it. Bubbles of gas at once escape from the tube at C showing that the pressure in A is tending to increase. If the jar is removed the stream of bubbles at once ceases and the liquid rises in the tube. The bulb B is sufficiently large to prevent liquid from reaching A. In the first part of this experiment the coal-gas passes more rapidly into the pot than the air inside can escape, so that the pressure rises. In the second part, the coal-gas

which has found its way into the pot diffuses out more rapidly through the walls than the air does inwards so that the pressure inside is reduced.

A similar experiment may be made using carbon dioxide instead of coal gas. For this purpose the apparatus is arranged as in Fig. 5·2 (*b*). A jar containing the carbon dioxide is placed so that there is an atmosphere of the gas round the porous pot. The liquid rises in the tube, showing that air is diffusing more rapidly from the pot than carbon dioxide is diffusing inwards. When the jar is removed, the pressure inside the apparatus increases and, depending on the relative amount of carbon dioxide which has entered the pot, a bubble of gas may escape from the tube immersed in the liquid.

The Diffusion of Solids.—Diffusion in solids has been investigated by SIR ROBERTS-AUSTEN, who placed an alloy of lead and gold (5 per cent. gold) in contact with a piece of lead, the two surfaces in contact being accurately plane and held together under pressure. The whole was heated at 165° C. for one month. On analysing various sections it was found that diffusion had taken place. The experiments were repeated at room temperature when it was observed that diffusion still occurred, only at a diminished rate.

The diffusion of one solid into another finds an important application in the 'cementation' process of converting iron into steel. The iron is placed in intimate contact with powdered carbon and then heated. The depth to which carburization takes place depends upon the temperature and time of heating.

Osmosis.—When red blood corpuscles are placed in water they expand rapidly and ultimately burst, but if they are placed in a strong salt solution they shrivel up. This phenomenon is characteristic of the membranes surrounding many animal and vegetable cells, for these allow water to pass through freely but retard or entirely prevent the passage of solids. ***Osmosis*** is the name given to this spontaneous passage of a liquid through a membrane. Its effects were first observed by the ABBÉ NOLLET in 1748, but it was left to a botanist, PFEFFER, to investigate it quantitatively. A piece of wet parchment paper is stretched over the end of a large thistle funnel and when nearly dry it is coated with glue along the boundary. The inverted funnel is partly filled with a solution of sodium chloride, cane-sugar, or some other substance, and immersed in water [cf. Fig. 5·3]. After standing for some time the level of the solution will have risen considerably; water must have passed through the parchment into the solution. This statement is not complete, for water will have passed from the solution into the water in the beaker at the same time as water passed from the beaker into the solution. This osmotic flow arises from the bombardment of the molecules upon the membrane; on the one side there are only molecules of water arriving at the membrane, whilst on the

other hand there are molecules of water and solute as well. Now such membranes are only slightly permeable to dissolved salts and the resultant effect is that more water molecules pass in one direction than the other.

An osmotic flow of the solvent is also observed when a membrane separates two solutions of the same nature but differing in concentration. The flow of solvent is such that the concentrations of the solutions tend to become equal, i.e. there is an excess of solvent passing from the weaker to the stronger solution.

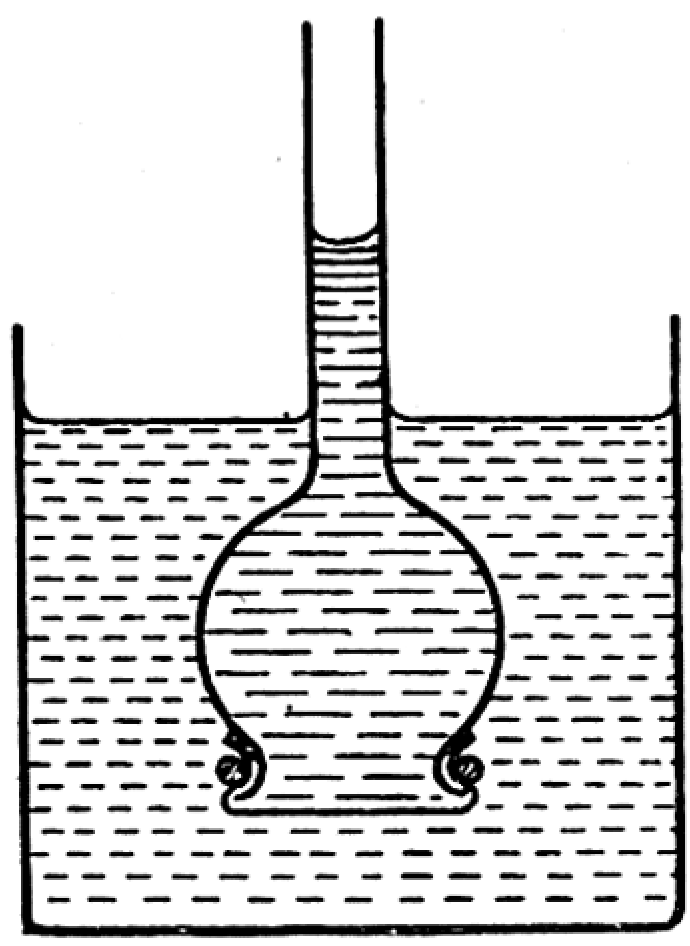


FIG. 5·3.—Osmosis.

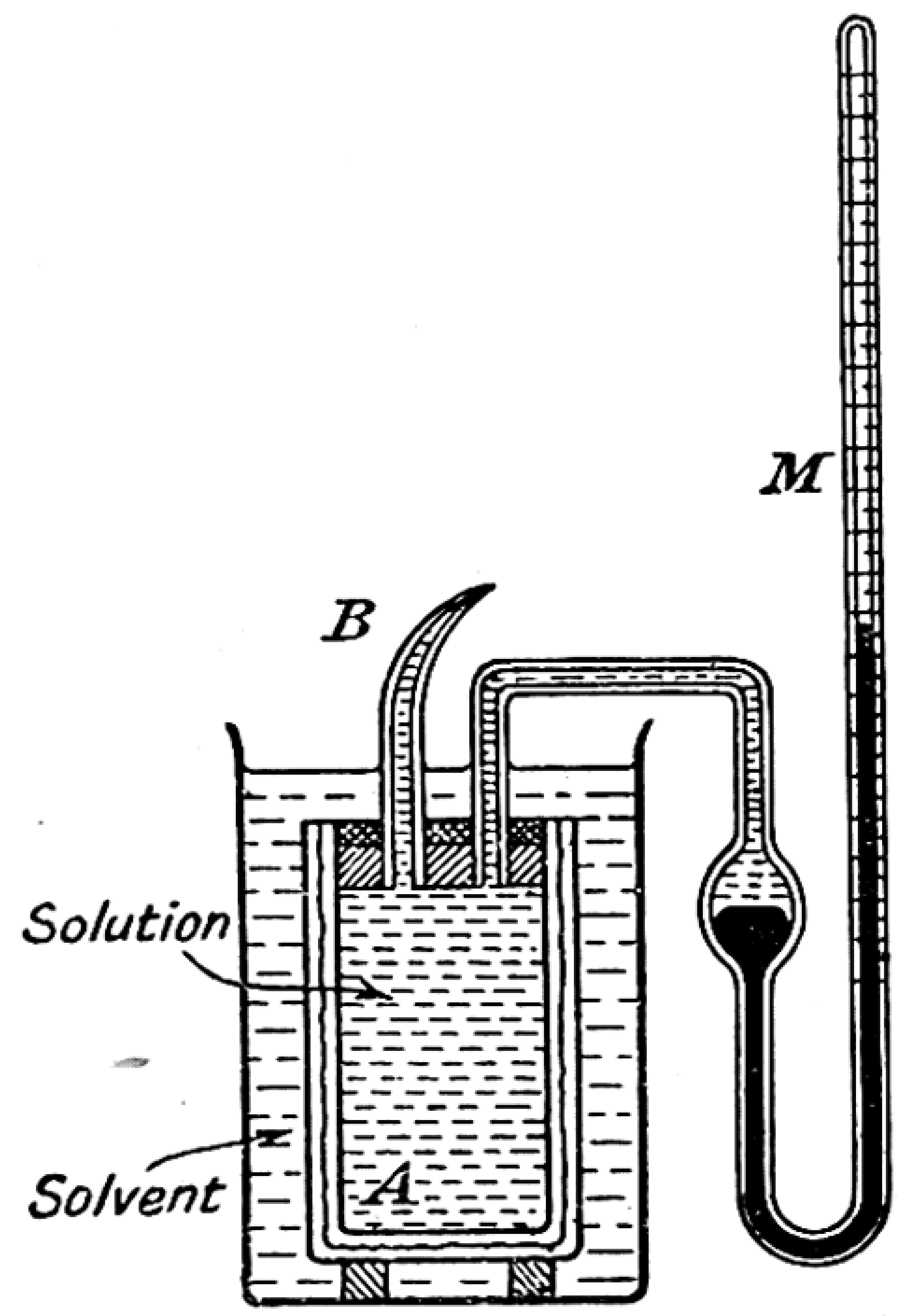


FIG. 5·4.—The Measurement of Osmotic Pressure.

Semipermeable Membranes.—A membrane which permits the solvent but not the solute to pass through it is termed a semipermeable membrane. One of the best-known membranes of this class is copper ferrocyanide.

Experiment. Place a weak solution of potassium ferrocyanide in the bottom of a beaker and when it has ceased to move introduce a strong solution of copper sulphate so that it lies below the ferrocyanide solution. A thin gelatinous precipitate of copper ferrocyanide is formed: it separates the two solutions. The membrane does not increase in thickness since the dissolved substances cannot pass through it, but after the lapse of about two hours it will be seen that the membrane has a distinct bulge upwards. This proves that more water passes downwards than flows upwards, and hence that the copper solution has the greater osmotic pressure.

Osmotic Pressure.—The membrane of copper ferrocyanide prepared in the above experiment is too fragile to support more than a

small pressure difference, but its strength is very considerably increased if it is produced in the walls of a porous pot. To prepare this membrane the porous pot is boiled in distilled water for several hours to remove air bubbles. A 0·25 per cent. solution of copper sulphate is then placed inside the pot and a 0·21 per cent. solution of potassium ferrocyanide outside. Each solution should reach very nearly to the top of the pot. Diffusion occurs and the two dissolved substances meet inside the walls of the pot where a membrane of copper ferrocyanide is formed. This process should be allowed to continue without interruption for two days. The pot thus prepared is boiled in several changes of distilled water and is then ready for use. If allowed to become dry it should be boiled for several hours to expel all air again.

If such a pot, provided with a rubber bung carefully waxed in position and provided with a long capillary tube, is filled with a saturated solution of cane sugar and then immersed in water, the change in level of the liquid in the capillary is very rapid. After several days a tube 1 mm. in diameter must be several metres long if the liquid is not to exude from it. This spontaneous differential flow of liquid through the membrane can be completely stopped by the application of a suitable pressure; the flow is reversed if the pressure is increased beyond this value.

Definition.—***That pressure which must be applied to a solution to prevent the spontaneous differential flow of liquid through a semipermeable membrane separating the solution and solvent is termed the osmotic pressure of the solution.***

To determine the osmotic pressure of a weak aqueous solution the apparatus shown schematically in Fig. 5·4 may be used. A mercury manometer, M, with one limb closed and containing air, or better, nitrogen, is connected to the porous pot, A, containing the solution. This solution is introduced through the tube B, which is afterwards hermetically sealed and the air in the connecting tubes displaced by some of the solution so that temperature changes do not affect the volume between the pot and the gauge. Water enters the solution and the pressure inside the pot increases. Ultimately this pressure ceases to change and this constant pressure is the osmotic pressure of the solution. It is calculated from the change in volume of the gas (air) in the closed limb of the manometer. The serious objection to this method lies in the fact that the water entering the solution changes the concentration of the latter so that the readings do not correspond to the osmotic pressure of the original solution: neither do they to the final solution, for its concentration is not uniform and it is the concentration of the solution in the immediate vicinity of the mem-

brane which determines the osmotic pressure which is measured. It is better to measure the external pressure which must be applied to the solution to prevent the passage of the solvent. Such methods must always be used for concentrated solutions. Lord Berkeley and Hartley have developed this method, but their apparatus is too complicated for a detailed description here.

The Fundamental Laws of Osmotic Pressure.—(*a*) At constant temperature the osmotic pressure of a dilute solution is directly proportional to the concentration of the solute in the solvent, i.e. it is inversely proportional to the volume of the solvent containing a given mass of dissolved substance.

(*b*) The osmotic pressure of a dilute solution is directly proportional to its absolute temperature.

The analogy between these two laws and those of Boyle and of Charles is very apparent: in fact, the osmotic pressure of a dilute solution is the pressure which the dissolved substance would exert if it existed as an ideal gas occupying the same volume and being at the same temperature as the solution.

The above laws apply to dilute solutions of non-electrolytes, but experiment shows that solutions of electrolytes have higher osmotic pressures than they would indicate. This is explained by the fact that such substances exist as ions when they are in solution.

The laws of osmotic pressure may be symbolized by the formula

$$pv = k\mathrm{T},$$

where p is the osmotic pressure, v the volume of solution containing 1 gm. of the solute, T is the absolute temperature, and k is a constant. van't Hoff showed that the constant k in the above 'osmotic equation' had the same value as the constant $\mathscr{R}$ in the characteristic equation for an ideal gas.

If one mole of a substance of molecular weight M is dissolved in a volume V cm.^{3}, then $\mathrm{V} = \mathrm{M}v$, so that the characteristic equation becomes

$$p\mathrm{V} = \mathrm{M}\mathscr{R}\mathrm{T}$$

Now it is found that $\mathrm{M}\mathscr{R}$ is a constant for all substances: it is denoted by R and is known as the universal gas constant. Thus

$$p\mathrm{V} = \mathrm{RT}$$

If Ω is the volume of a solution containing N moles of dissolved substance, $\mathrm{NV} = \Omega$, so that

$$p\left(\frac{\Omega}{\mathrm{N}}\right) = \mathrm{RT}.$$

If $C = \frac{N}{\Omega}$, the concentration in mole. cm.$^{-3}$, then

$$\frac{p}{C} = RT.$$

If c is the concentration in gm. cm.$^{-3}$, $c = MC$, so that

$$\frac{p}{c} = \frac{RT}{M}.$$

Thus, if the osmotic pressure, in absolute units, of a solution at temperature T and concentration c is known, it is possible to determine the molecular weight of the dissolved substance.

Osmotic Pressure and the determination of Molecular Weight.—It is customary in experimental work on osmosis to measure the pressure in atmospheres and to consider the volume in cm.^{3} occupied in solution by 1 gram-molecule (or 1 mole) of the dissolved substance. The characteristic equation for an ideal gas, when the pressure is measured in atmospheres and 1 gram-molecule occupying a volume V is considered, then becomes

$$PV = \overline{R}T,$$

and $\overline{R}$ is a universal constant for all gases. It must be noted, however, that $\overline{R}$ is different from the universal gas constant R which appears in the ideal gas equation $pV = RT$, where the pressure p is expressed in absolute units. It is known that 1 gram-molecule of a gas at S.T.P. occupies 22,415 cm.^{3}. Hence

$$1 \times 22{,}415 = \overline{R} \times 273,$$

$$\text{or } \overline{R} = 82{\cdot}06 \text{ cm.}^3 \text{ atmos. deg.}^{-1} \text{ K. mole.}^{-1}$$

This enables us to calculate a value for the osmotic pressure of a non-electrolyte in solution or, knowing the osmotic pressure, to determine a value for the molecular weight of the dissolved substance. Let m gm. of a substance of molecular weight M be dissolved in 100 cm.^{3} of water at $\theta°$ C. Then the number of gram-molecules in this volume is $\frac{m}{M}$, so that 1 gram-molecule would occupy $\left(\frac{M}{m} \times 100\right)$ cm.^{3}. Let P be the required pressure in atmospheres. Then

$$P \times \left(\frac{M}{m} \times 100\right) = \overline{R} \times (273 + \theta)$$

$$\therefore\ P = \frac{0{\cdot}821 \times (273 + \theta) . m}{M} \text{ atmosphere.}$$

$$= \frac{0{\cdot}821 m(273 + \theta)}{M} \text{ atmosphere.}$$

Isotonic Solutions and Plasmolysis.—If the variation with concentration of the osmotic pressure of one aqueous solution at a constant temperature has been investigated experimentally, the unknown osmotic pressure of another aqueous solution may be found in the following way. The strength of the first (or standard) solution must be adjusted until its osmotic pressure equals that of the unknown solution; the two solutions are then said to be ***isotonic*** with each other. To carry out such an experiment a convenient semi-permeable membrane must be available. DE VRIES, a Dutch botanist, in 1888 used the cells of the leaves of certain plants, among which he mentions those of *Tradescantia discolor*, *Begonia manicata*, and *Curcuma rubricaulis*. Such cells consist of a mass of protoplasm (living matter) containing sap vacuoles separated from the protoplasm by the so-called ***inner plasma membrane,*** and surrounded by a cellulose wall which is sufficiently

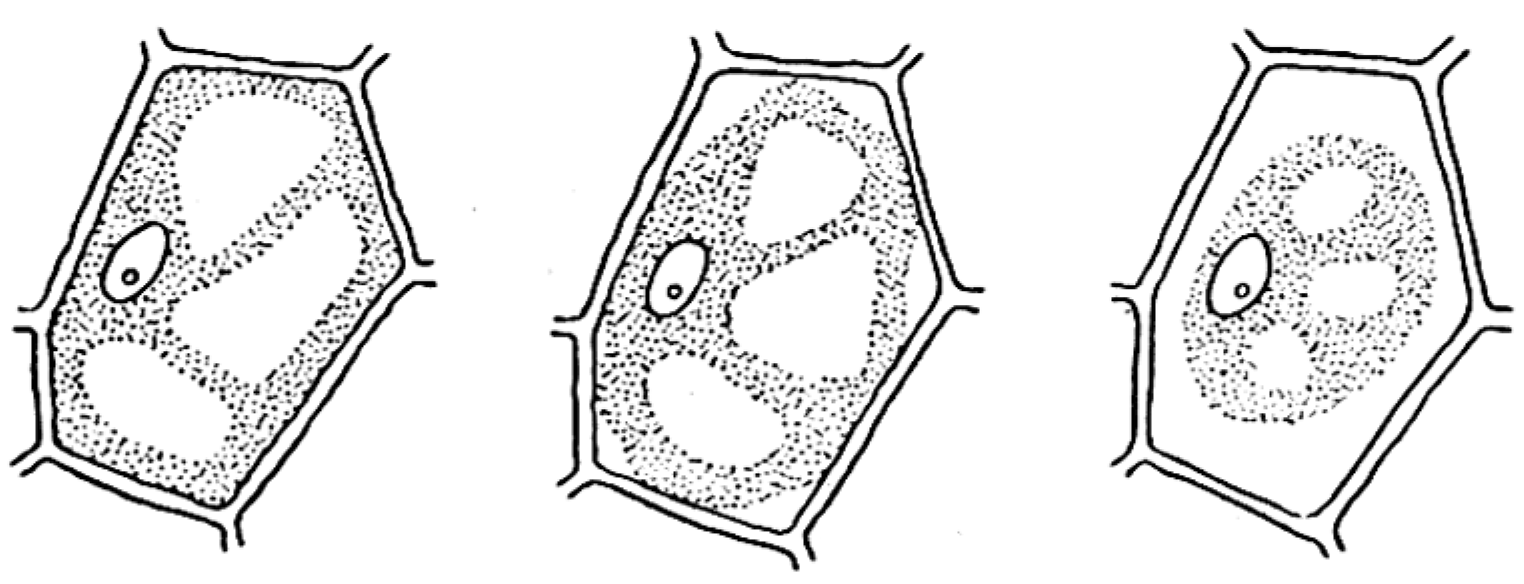


FIG. 5·5.

strong to withstand forces tending to change its shape, i.e. its changes in form are only minute. This wall is separated from the general mass of protoplasm by the ***outer plasma membrane.*** The whole of the contents of such a cell is termed the ***protoplast.***

Now the vacuoles contain the cell sap in which dissolved substances exist. It is not known for certain whether or not the plasma membranes are the controlling semi-permeable membranes through which water passes to the vacuoles, or whether the whole lining of protoplasm acts in this way. The latter assumption is generally adopted as it simplifies the discussion.

If, therefore, these cells are immersed in a solution having an osmotic pressure equal to their own, the cell, viewed under a microscope, will present its normal appearance [Fig. 5·5 (*a*)]. If the cell is placed in a solution having a greater osmotic pressure than its own, water will pass from the cells into the solution; the protoplast will shrink and the cell will appear as in Fig. 5·5 (*b*), or finally as in Fig. 5·5 (*c*). If the cell is placed in a solution the osmotic

pressure of which is less than its own then water will pass into the cell, but this will only be very slightly distended on account of the relatively strong cellulose wall which forms the external boundary of the cell.

In order to find a solution which shall have an osmotic pressure equal to that of the cell, experiments are first made with a solution having an osmotic pressure greater than that of the cell. The solution is then diluted gradually until the protoplast just maintains its normal form. When this occurs the solution in the cell and the one in which the cell is immersed, each exert the same osmotic pressure, i.e. they are ***isotonic*** with one another. The above method of determining osmotic pressure either of the solution in the vacuole of a leaf, or of an unknown aqueous solution, is referred to as the ***plasmolytic*** method.

Dialysis.—In his famous researches on the phenomenon of diffusion, GRAHAM found that some substances (mineral acids and salts), the so-called crystalloids, were able to pass through certain semi-permeable membranes. The other type of substance (gum, for example) is known by the name of colloid. The line of demarcation between the two types is not sharp, some substances behaving like crystalloids or colloids according to the nature of the solvent in which they are dissolved. The classical example is that of sodium stearate, $C_{17}H_{35}.COONa$, which acts as a colloid when an aqueous solution is made, whereas it exhibits the properties of a crystalloid when in alcoholic solution. Crystalloids are such that when they are dissolved in water, they produce a diminution of its saturation vapour pressure, a fact which is revealed by the lowering of the freezing-point and the raising of the boiling-point of the water; on the other hand, colloids produce no appreciable effect. Whenever a colloid is made it almost invariably contains a quantity of the crystalloid from which it has been prepared. The separation of these substances is carried out by means of a process known as 'dialysis.' The mixture is placed in a cylindrical vessel, the bottom of which consists of parchment paper. The whole is placed in a liquid medium capable of dissolving the crystalloid. The crystalloid diffuses through the membrane until the concentration of this substance is the same on both sides of the medium. Frequent renewals of the solvent are therefore made, and in this way a colloid, free from crystalloids, is obtained.

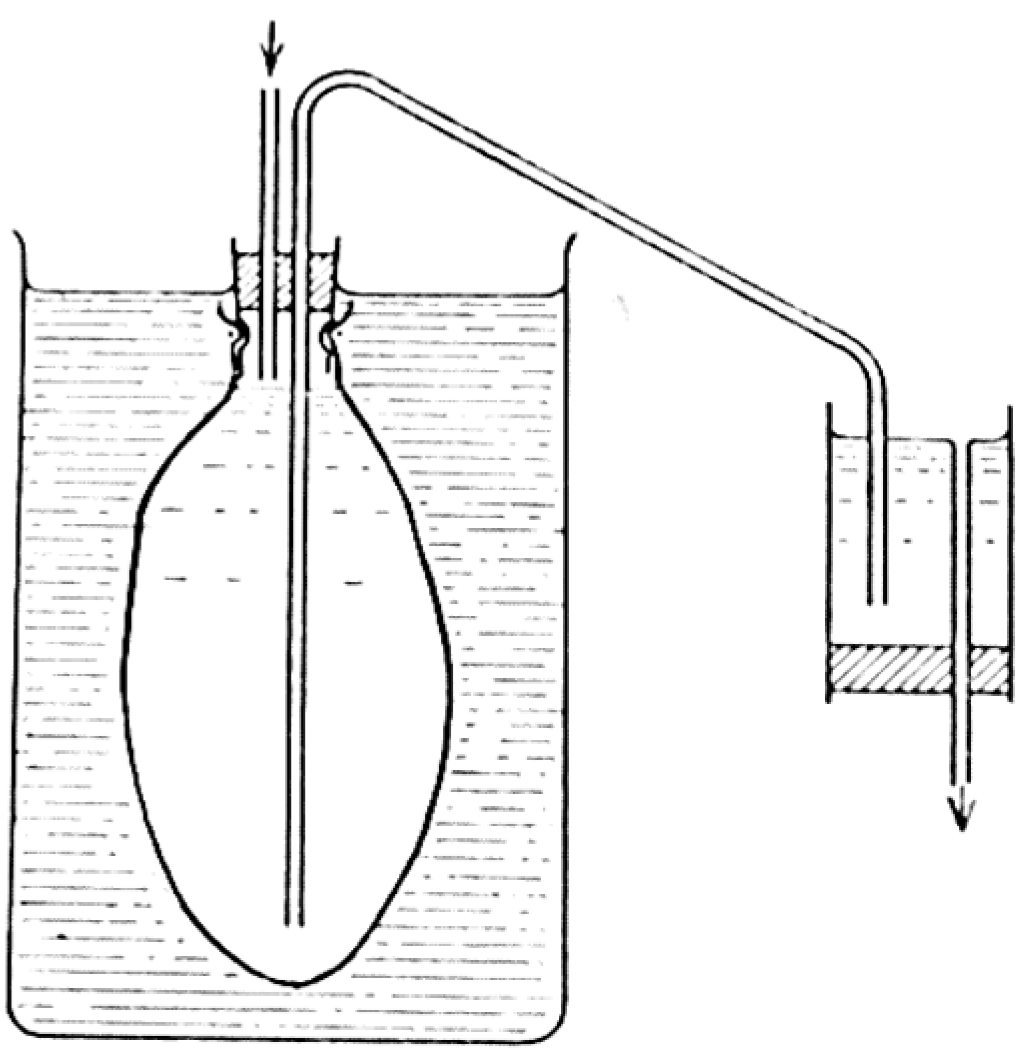


FIG. 5·6.—Apparatus for use in 'Hot Dialysis.'

The membranes which are used for dialysis are gold-beaters' skin, fish bladder, and parchment paper. The speed at which dialysis takes

place rapidly increases with rise in temperature, and in order to effect this increase the hot dialyser shown in Fig. 5·6 may be employed. The colloid from which the crystalloid is to be removed is placed in a two-litre beaker. This is heated with the aid of a suitable burner. A membrane is attached by string to a cork suitably bored and fitted with two glass tubes to allow distilled water to pass into the bag which the membrane forms, any excess being removed by means of the automatic device indicated. This excess water carries with it the crystalloid which has passed through the membrane.

Surface Energy and Surface Tension.—Everyone will have noticed that when a small amount of liquid is brought into close contact with a solid, the liquid either spreads itself over the surface of the solid, or else collects itself into small drops, and that most liquids tend to rise in capillary tubes to a distance above the surface of the liquid in the containing vessel, whereas some, such as mercury and molten metals, act in an exactly opposite way. To explain these phenomena it has sometimes been maintained that the surface of a liquid must be endowed with some peculiar property, e.g. the surface may be skin-like. LANGMUIR and N. K. ADAM have shown that all these properties of liquids can be attributed to molecular happenings inside the liquid. The hypothesis that the surface of a liquid has a skin-like structure has been superseded by these more modern views. The fact that the molecules of a liquid are free to move has been confirmed by experiments on Brownian movement. These molecules must be very closely packed together, for experiment has shown that a liquid resists forces tending to compress it, even if the forces are enormously large. Since the molecules are so close together, the forces of attraction between neighbouring molecules in liquids must be very large. When, however, a molecule is at the surface of the liquid it will not be attracted equally in all directions, for there is no liquid above it. In consequence of this such molecules will tend to move towards the interior of the liquid. Since the molecules occupy space, i.e. there is a definite number per unit volume, the surface tends to diminish in area. In support of these remarks we have the fact that liquids tend to assume that shape which has the minimum area for a given volume. If a drop is subject to other forces comparable with those discussed above its shape will be slightly distorted from that having the minimum area, e.g. a rain-drop hanging from a window-pane.

In virtue of these forces, directed inwards, molecules at the surface will possess a certain amount of energy due to position. The amount of this energy per unit area is termed the ***surface energy***. The surfaces of both liquids and solids possess surface energy but it is only when the surface is mobile that its effects become apparent. The fact that a liquid surface is the seat of

potential energy manifests itself very vividly when a soap film is ruptured (with the aid of a pointed piece of filter paper, for example), for the liquid is projected in all directions with a considerable velocity, i.e. the potential energy has been converted into kinetic energy.

Let a liquid film be formed between two limbs of a bent wire, BAC, Fig. 5·7, and a horizontal straight wire, XY, placed across them. Suppose that a force, F, acting normally to XY is necessary to maintain equilibrium when the film is vertical. Then F must be balanced by a force on the wire due to the film. Suppose γ is the magnitude of this force per unit length of the wire. If the length of XY is l, the total force on the wire from the above cause is $2\gamma l$, the factor 2 being introduced since the film has two sides. Hence

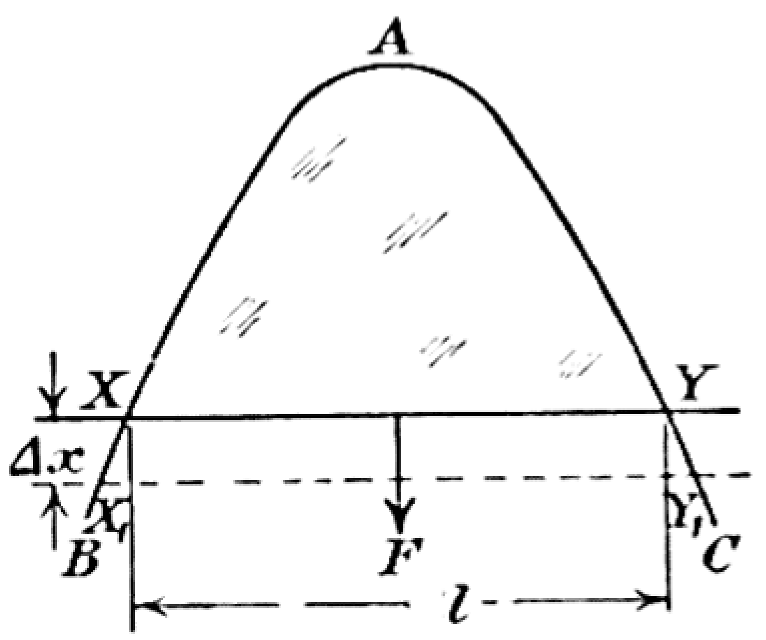


FIG. 5·7.—Surface Tension.

$$F = 2\gamma l.$$

γ is termed the ***surface tension*** of the liquid.

[It should be noted that if parallel wires are used for the purpose of forming a film between them, the system is unstable. For example, if F, is too large, the force $2\gamma l$ never becomes sufficient to balance F for l remains constant. The instability does not matter as far as theory is concerned, but with the stable arrangement here adopted a rough estimate of γ may be made. If the weight of the wire is not sufficient it may be loaded. Then $F = mg$, where m is the total mass of the wire and its load.]

On the Relation between Surface Energy and Surface Tension.—Again consider Fig. 5·7. Let XY move through a small distance δx to a parallel position X_1Y_1 ,the external force on the wire being F. Now when a film is stretched in this way its temperature falls unless heat (thermal energy) is communicated to it. We shall suppose that the heat necessary to restore the film to its original temperature has been supplied.

If ε is the surface energy of the film, i.e. the potential energy per unit area of the surface, the increase in potential energy of the 'surface' molecules is $(2l.\delta x)\varepsilon$, the factor 2 being introduced since the film has two surfaces. The work done by the stretching force is $F.\delta x$. These two quantities cannot be equated, however, for heat has been communicated to it from external bodies. If δQ is the heat (thermal energy) supplied to restore the temperature of the film to its original value, we have

$$(2l.\delta x)\varepsilon = F.\delta x + \delta Q.$$

Now the force F is equal and opposite to the pull of the film on the wire XY, when the film is in equilibrium. If γ is the pull per unit length, then $2l.\gamma = F$, the above equation becomes

$$2l.\delta x.\varepsilon = 2\gamma l.\delta x + \delta Q,$$

or
$$\varepsilon = \gamma + \left(\frac{\delta Q}{2l.\delta x}\right).$$

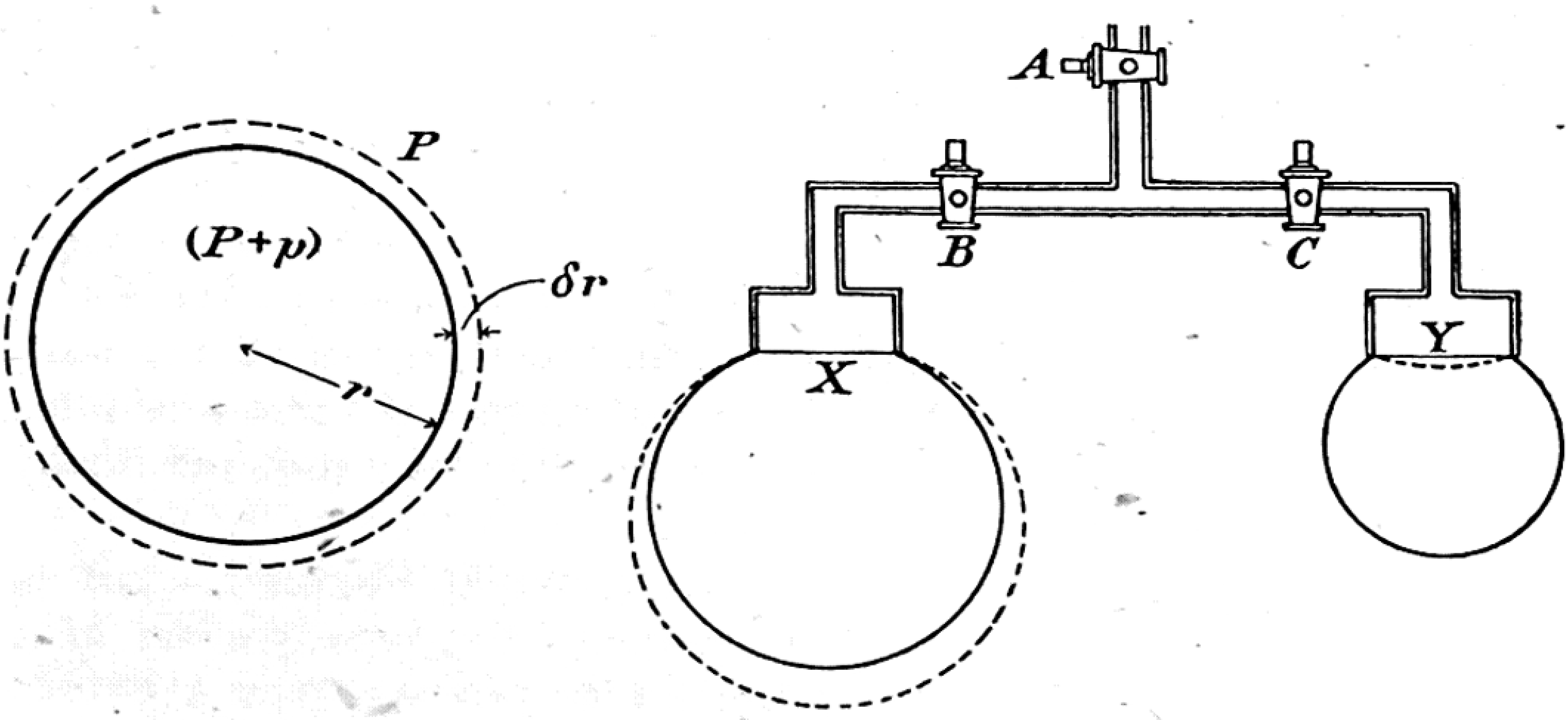
This may be written

$$\varepsilon = \gamma + \eta,$$

where $\eta = \dfrac{\delta Q}{2l.\delta x}$, the thermal energy supplied per unit increase in area of the film.

Now the force γ exerted on each unit length of the wire is called the ***surface tension*** of the liquid and the above shows that the intrinsic surface energy of a liquid is really the sum of two quantities—a 'thermal' part denoted by η, and a 'mechanical' part γ, or $\varepsilon - \eta$; we see, therefore, that the surface tension is equal to the 'mechanical' part of the surface energy. HELMHOLTZ called this 'mechanical' part of the surface energy the free energy of the surface, or the surface free energy. It will be noted also that the increase in the total free surface energy of a surface is equal to the external work done on that surface, providing heat is supplied to keep its temperature constant.

The Pressure Difference across a Spherical Surface.—Let r, Fig. 5·8 (a), be the radius of a spherical bubble of gas in a liquid. Let P be the pressure outside the bubble. We have to show that



(a) FIG. 5·8. (b)

the pressure inside is equal to $P + p$, where p is a quantity to be determined. For this purpose let r become $r + \delta r$, where δr is a very small quantity, in fact so small that the pressure inside is

not altered thereby. Moreover, let heat be supplied to the film so that its original temperature is restored. The area of the curved surface has increased from $4\pi r^2$ to $4\pi(r + \delta r)^2$. If γ is the surface tension of the liquid or, as we have just seen, its free surface energy per unit area, the increase in free surface energy is $4\pi\gamma[(r + \delta r)^2 - r^2] = 8\pi\gamma r \,.\, \delta r$, since $(\delta r)^2$ may be neglected. This is equal to the work done in expanding the bubble. Since pressure is defined as the force per unit area, the total force acting on the inner surface of the bubble is $4\pi r^2(\mathrm{P} + p)$, while that on the outer surface is $4\pi r^2\mathrm{P}$. Since these forces are opposed to one another the net work done on the film is $4\pi r^2 p \,.\, \delta r$. Equating the two expressions obtained for this work, we have

$$4\pi r^2 p \,.\, \delta r = 8\pi\gamma r \,.\, \delta r,$$

or

$$p = \frac{2\gamma}{r}.$$

If the bubble had been a soap bubble this excess pressure would have been $\dfrac{4\gamma}{r}$, for a soap film has a double surface.

The fact that the pressure inside a soap bubble diminishes as the radius increases is shown by the following experiment. Two brass cups, X and Y, Fig. 5·8 (*b*), about 2 cm. in diameter and 1 cm. long, are connected to stop-cocks A, B, and C as shown. The open ends of X and Y are immersed in a soap solution and soap bubbles differing considerably in diameter blown. B is open and C closed while the larger bubble is being formed, and vice versa. A is then closed and the two bubbles placed in communication with each other by opening the stop-cocks B and C. Air passes from the smaller bubble into the larger one, causing the latter to expand and the former to shrink. This process continues until the radius of curvature of the larger bubble is equal to the radius of curvature of the soap film which finally protrudes below the open end of Y and which is a portion of a spherical surface—see the dotted outlines on the diagram. After a time the thickness of the walls of the large bubble become so thin that it bursts: the film remaining on Y at once becomes flat, and after some time very thin and finally breaks.

Pressure Difference across a Cylindrical Surface.—Let us now assume that Fig. 5·8 (*a*) represents the cross-section of a cylindrical bubble. Since it is difficult to produce such a bubble in a liquid we will assume that it consists of a soap film having *two* surfaces. Consider a length l of this cylinder. When r becomes $r + \delta r$, as before, the increase in area is $2[2\pi(r + \delta r - r)l]$. Let thermal energy be supplied to the film so that its temperature assumes its original value. The increase in the free surface energy

is $4\pi l\gamma \,.\, \delta r$. Now the work done, due to the pressure difference p, is $2\pi r l p \,.\, \delta r$. Equating these two quantities we have $p = \frac{2\gamma}{r}$. When there is only one cylindrical surface the excess pressure is $\frac{\gamma}{r}$.

Angle of Contact.—If a piece of clean glass is inserted into water so that it is in a vertical position, it will be found that the liquid near the glass has been drawn some distance beyond the level of

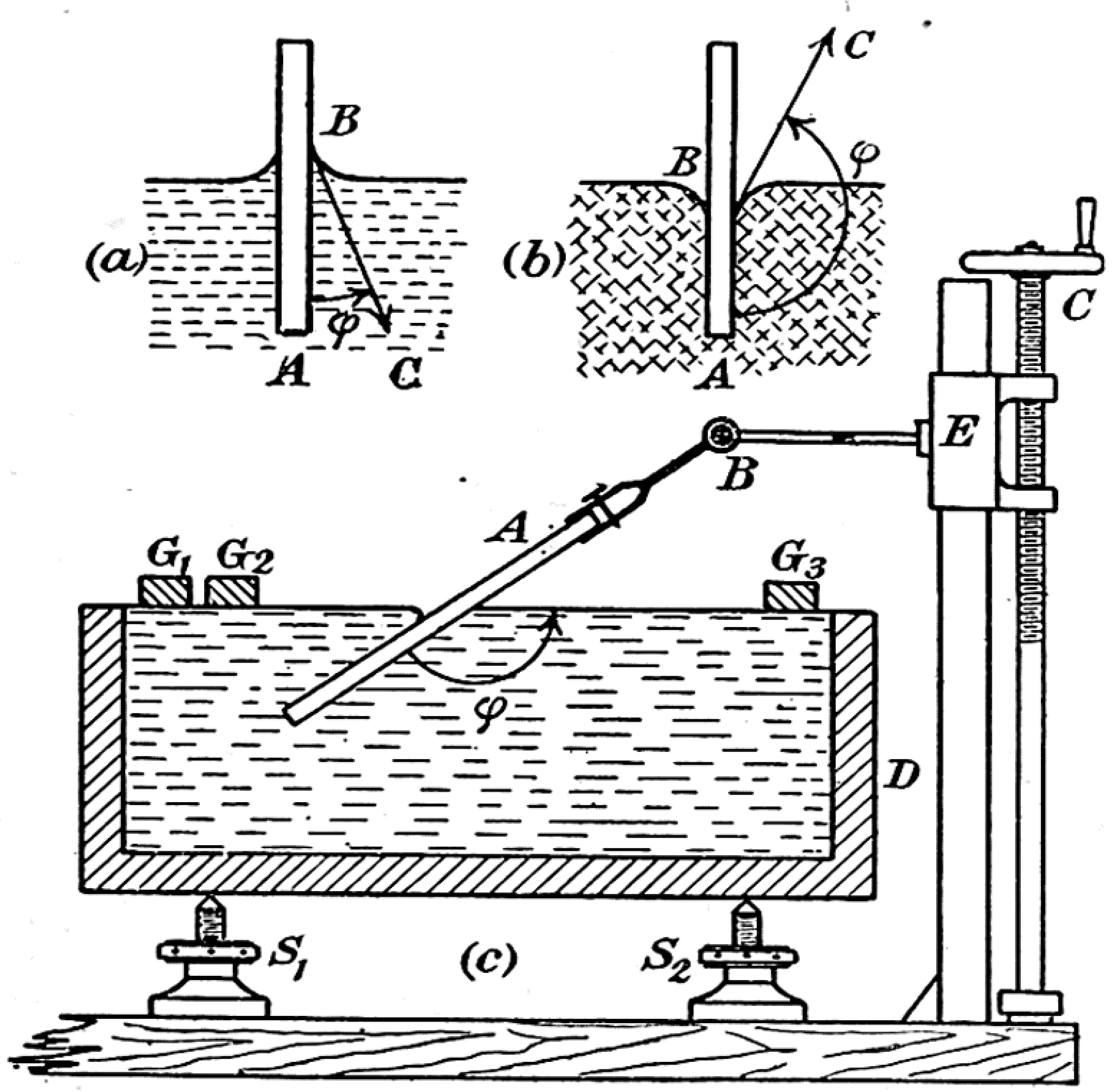


FIG. 5·9.—Angles of Contact and their Measurement.

the rest of the water. The $\widehat{ABC}$, Fig. 5·9 (*a*), i.e. the angle between the solid surface in the water and the tangent to the water surface where it meets the glass, is called the angle of contact for a water-glass interface. For water in contact with glass this angle is very small, whilst for benzol in contact with glass it is zero.

When the above experiment is repeated with mercury the liquid near the glass is depressed below the general level of the mercury surface. The angle of contact is again ABC, Fig. 5·9 (*b*), but it is now quite large (approximately 135°). It should be noted that, although the surface tensions of two liquids may be equal, they may not exhibit the same capillary phenomena, for their angles of contact with a given material may be different.

The effect of the angle of contact on the shape of a small quan-

tity of liquid placed on a flat surface is easily shown as follows :—Water placed on a clean glass surface spreads itself over the glass, but if water is similarly placed on a greased plate it remains as a 'drop.' Traces of dirt or grease alter the angle of contact very considerably ; that is the reason why rain water persists as a drop when it alights on a window-pane, for such a piece of glass is never chemically clean.

To determine the angle of contact between water and glass coated with paraffin wax, N. K. Adam used an apparatus similar to that shown in Fig. 5·9 (c). A is a section of the plate at right angles to its faces. It is held in a clamp which may be rotated about a horizontal axis through B. The clamp may be moved vertically by means of the screw C, and the carriage E which it operates.

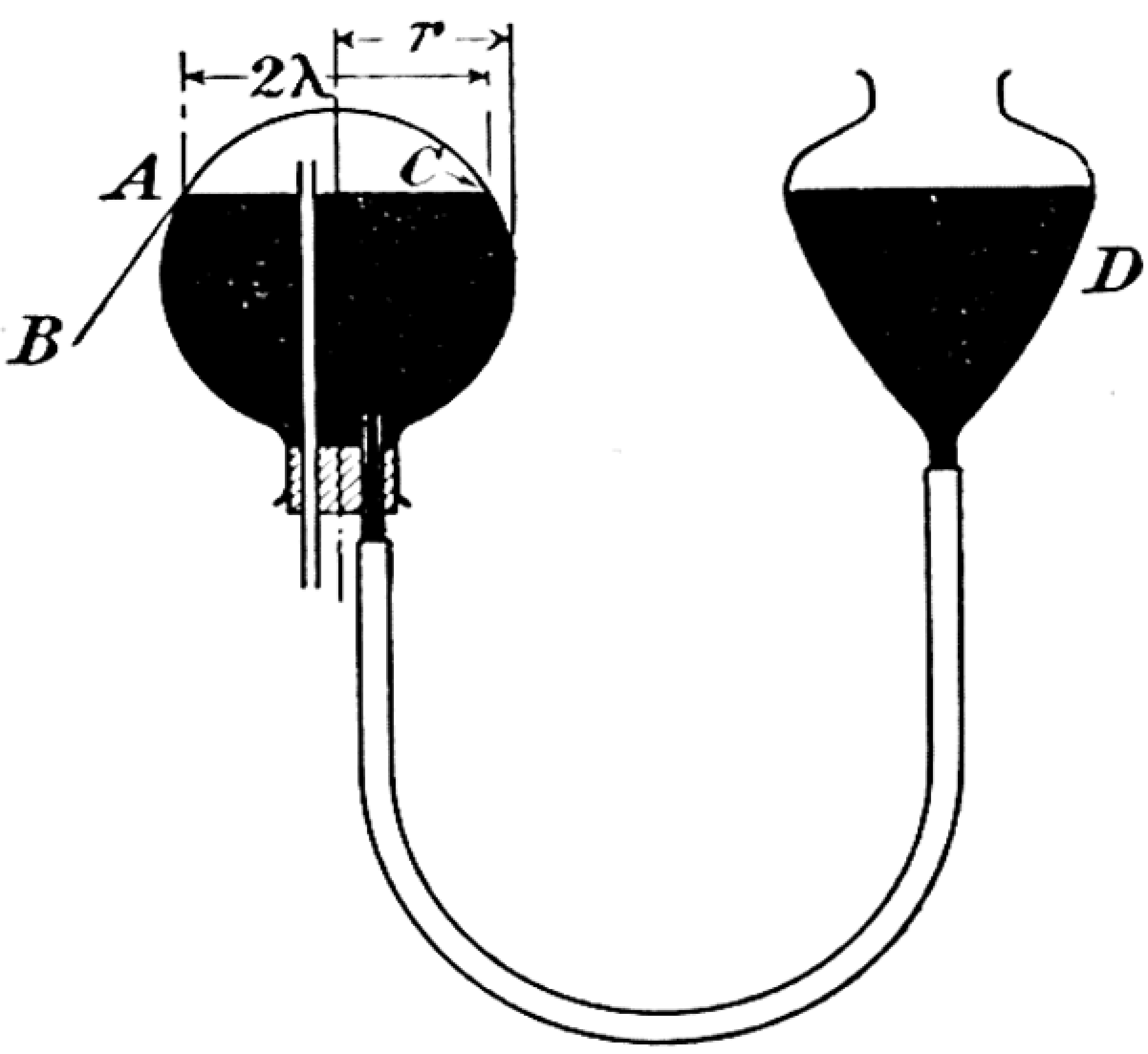


Fig. 5·10.—Angle of Contact of Mercury with Glass.

D is a glass trough, coated inside with paraffin wax so that it may be filled with water above the level of its sides which have been ground flat on the top. This surface is made horizontal with the aid of the screws S_1 and S_2. G_1, G_2 and G_3 are rectangular pieces of glass coated with wax and resting on the sides of the trough, and in contact with the liquid. By moving G_1 and then G_2 across from the right-hand side of the trough to the positions indicated, the surface of the liquid is freed from contamination. The plate is set in turn at various angles of inclination until a position is found for which the water-surface on one side of the plate remains undistorted right up to the line of contact with the solid. If ϕ is the angle between the trace of the plate and the undistorted surface of the water (as measured with the aid of a protractor), then ϕ is the angle of contact required.

In actual practice it is found that ϕ depends on whether the plate A is being pushed into the water or raised. This effect is easily observed by using the rack and pinion to impart the necessary vertical motions to the plate, and the corresponding angles of contact measured in the usual way. If ϕ_1 and ϕ_2 are the 'advancing' and 'receding' angles of contact, it may be shown that $\phi = \frac{1}{2}(\phi_1 + \phi_2)$.

An interesting method for investigating the angle of contact between mercury and glass is as follows :—The level of some mercury in an inverted spherical flask is adjusted by raising or lowering the reservoir D, Fig. 5·10, until the mercury surface in the flask is plane at points where it meets the glass. The angle BAC $= \phi$ is the required angle of contact. If 2λ is the length AC, and r the radius of the flask, $\phi = \sin^{-1}\frac{\lambda}{r}$: it must be remembered that $\frac{\pi}{2} < \phi < \pi$.

Liquid in Contact with a Solid.—We now have to account for the fact that the surface of a liquid near its place of contact with a solid is, in general, curved, even when gravity is the only external force acting throughout the mass of the liquid. Let ABC, Fig. 5·11, be the surface of the liquid. Consider the forces acting on a molecule M in the surface of the liquid and near to the solid D. They are :—

(i) its weight acting vertically downwards ;

(ii) the attraction of the solid on M, the direction of which will be along that normal to the surface of the solid which passes through M (since M is very close to the solid) ;

(iii) the force arising from the attraction of neighbouring liquid molecules. This will be directed towards the interior of the liquid.

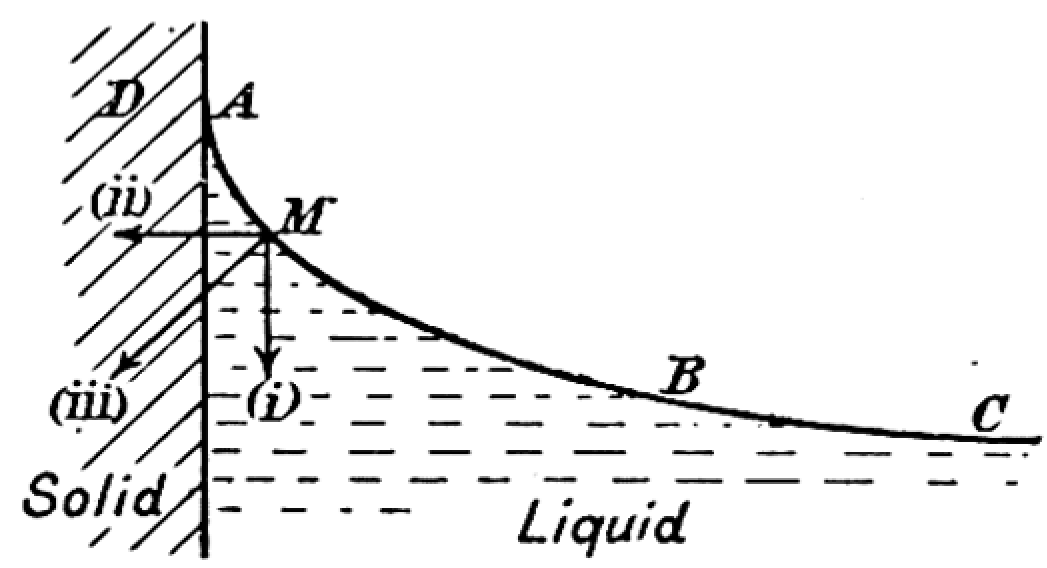


FIG. 5·11.—Liquid in Contact with a Solid.

Now the resultant force exerted on a molecule in an ideal liquid at its free surface must be normal to the surface. Hence the normal to the liquid surface at M will be determined by the resultant of the above three forces. In general, this resultant does not act along (i), i.e. the surface of the liquid at M is not horizontal.

For a molecule near C, a point at a considerable distance from the solid, the only finite forces are (i) and (iii) and these then act vertically downwards, i.e. the surface is flat.

For molecules at B, for example, there is a finite force (ii) but less than the force (ii) on M ; in consequence, the surface is more nearly flat.

The Rise of a Liquid in a Capillary Tube.—For the sake of simplicity we shall first assume that the angle of contact is zero. Let AC, Fig. 5·12 (*a*), be the surface of a liquid in a capillary tube of radius r. We assume that AC is part of a sphere of radius r. The pressure over the curved surface is everywhere atmospheric. At B, a point just below the surface and therefore in the liquid,

the pressure is less than atmospheric by an amount $\frac{2\gamma}{r}$ [cf. p. 116]. At D, a point below B and lying in the same horizontal plane as the surface of the liquid outside the tube, the pressure is atmospheric. Now the difference in pressure between the two points B and D is equal to the pressure exerted by a column of liquid of height DB$=h$ (say). If ρ is the density of the liquid, this difference is $g\rho h$. The pressure at B is therefore less than atmospheric by this amount. But it has already been shown that this difference is $\frac{2\gamma}{r}$. We therefore have

$$\frac{2\gamma}{r} = g\rho h.$$

Now suppose that the angle of contact between the liquid and the material of the tube is ϕ—cf. Fig. 5·12 (*b*). Let R be the radius of curvature of the liquid surface at its lowest point—if the bore of

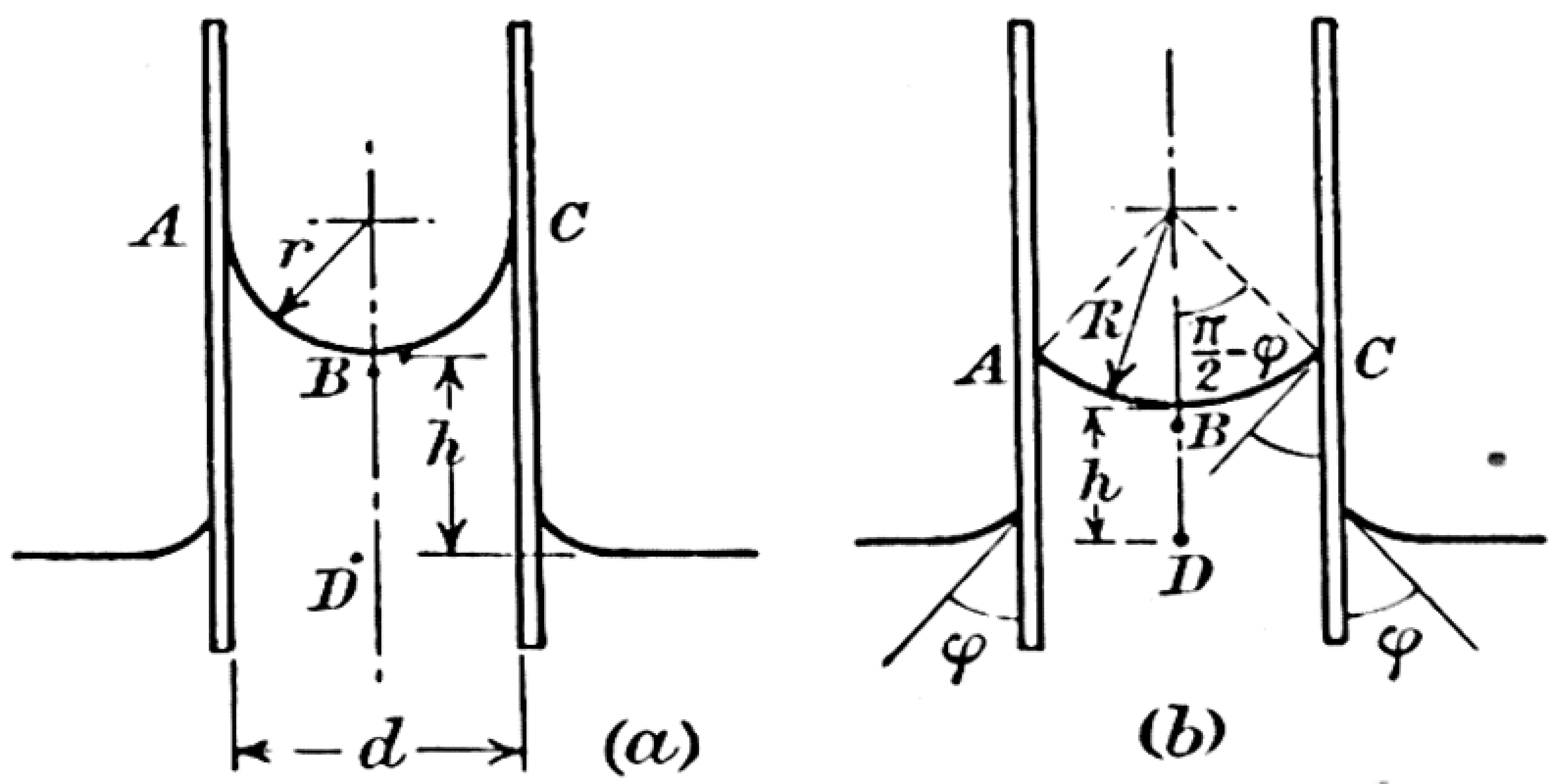


FIG. 5·12.—Rise of Liquids in Capillary Tubes.

the capillary is small, R is constant at all points on the liquid surface. Then, as before, if Π is the atmospheric pressure,

$$\text{Pressure at B} = \Pi - \frac{2\gamma}{\text{R}}.$$

But pressure at D $= \Pi =$ pressure at B $+ g\rho h$.

$$\therefore \frac{2\gamma}{\text{R}} = g\rho h.$$

But $r = \text{R} \cos \phi$; therefore $\frac{2\gamma \cos \phi}{r} = g\rho h$.

It should be mentioned, perhaps, that if ϕ is finite, values of the surface tension of a liquid deduced from measurement of its rise in capillary tubes are unreliable, since the magnitude of ϕ is always uncertain; moreover, ϕ varies considerably with the degree of contamination of the surfaces in contact. The above theory is necessary, however, for academic purposes.

The Rise of a Liquid between Vertical Plates.—(*a*) ***Parallel plates.*** To calculate the amount of this rise we may use Fig. 5·12 (*a*). Let the vertical lines in that diagram now represent sections of the two parallel plates at distance d apart. We assume AC to be a section of a cylindrical surface of diameter d so that the pressure at B is less than atmospheric by an amount $\frac{\gamma}{r}$, or $\frac{2\gamma}{d}$, since $d = 2r$. Proceeding as before we obtain (if the contact angle is zero),

$$\frac{2\gamma}{d} = g\rho h.$$

(*b*) ***Inclined Plates.***—Fig. 5·13 represents two vertical glass plates, AOB and OAD, inclined to one another at a small angle θ. When these are inserted in a liquid the latter rises between the plates. To determine the shape of the curve in which AOC, the vertical plane through OA and bisecting the angle θ, i.e. the plane of co-ordinates, intersects the liquid surface, consider an element PQR of the surface at right angles to the intersection of the liquid surface with the plane AOC. Let (x,y) be the co-ordinates, referred to OC and OA as axes, of Q the middle point of the element PQR. [$P_1Q_1R_1$ is another such element. Notice that the projections p, q, and r of the points P, Q, and R respectively, on the horizontal plane through Ox do not lie in a straight line.] Then if the liquid wets the glass, the surface at PQR is part of that of a cylinder whose diameter is equal to the distance between the plates at Q. This distance is $x\theta$, since θ is small.

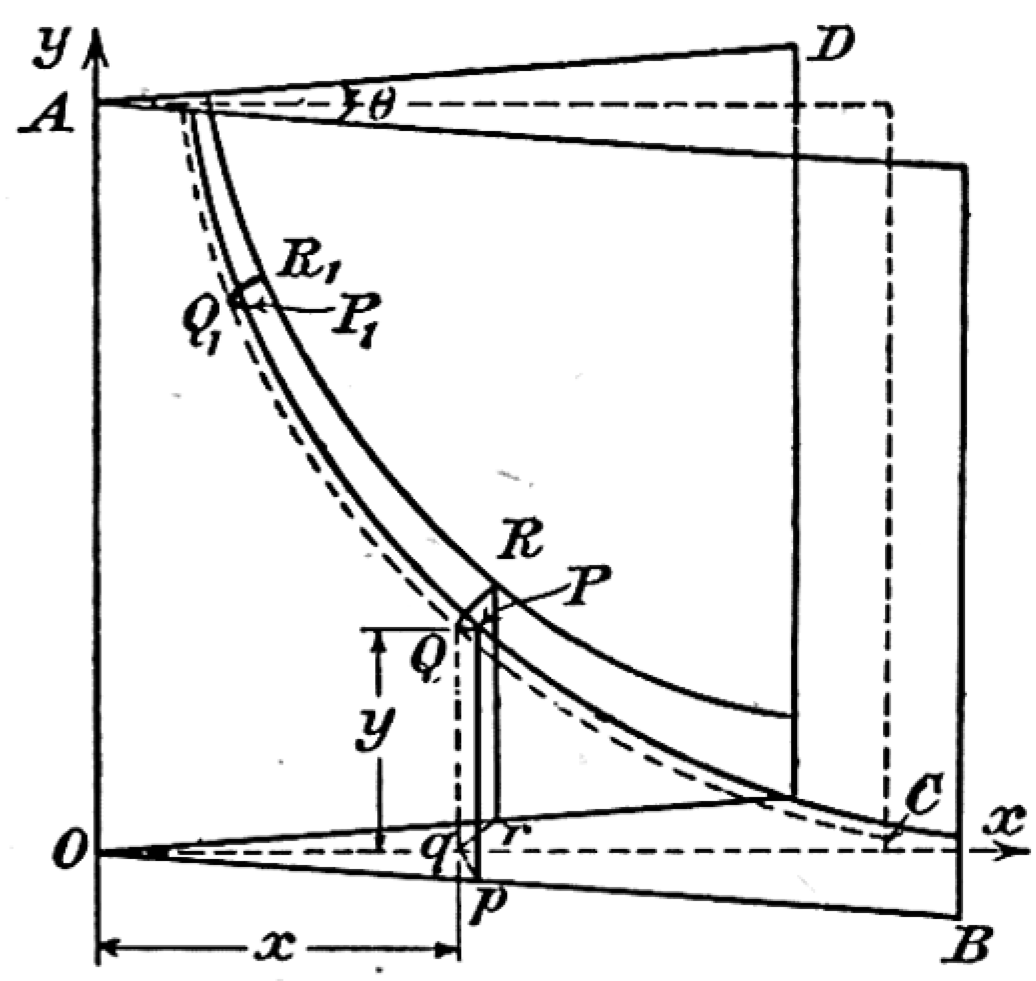


FIG. 5·13.—Rise of a Liquid between inclined Vertical Plates (end effect neglected).

The height y to which the liquid rises is therefore given by

$$g\rho y = \frac{\gamma}{\frac{1}{2}x\theta},$$

i.e. $xy = 2\gamma/g\rho\theta =$ constant. The surface is therefore part of a hyperbola, whose asymptotes are the axes of co-ordinates.

Experimental Determination of Surface Tension.—(*a*) ***Rise in a Capillary Tube Method.*** Select a piece of glass tubing about 0·4 cm. diameter and heat it in a bunsen flame, rotating the tube all the time. When the glass begins to soften, apply a gentle pressure along its length so that the walls of the tube thicken.

Then remove the glass from the flame and ***slowly*** pull the ends apart. The capillary tube thus constructed is clean, a condition which is absolutely essential if a reliable value for γ is to be obtained. When the tube is cold select a length from the centre of the drawn-out portion and attach to it a very thin glass rod, R, drawn out to a point and bent twice at right angles as in Fig. 5·14. Bands B_1 and B_2 cut from a length of rubber tubing enable this rod to be attached to the tube easily.

Now clamp the capillary A in a vertical position and place the liquid whose surface tension is to be measured below the tube so that the latter is immersed to a greater depth than that at which it is to be used and then raise it slightly. If the liquid falls back readily as the tube is raised we may assume that the tube and liquid are not contaminated. Continue to raise the tube until the end of the rod is just about to break through the liquid surface. To measure the height of the liquid in the capillary a vernier microscope, M, should be used. The microscope is focussed on the lowest point of the liquid surface in the capillary and the reading on its scale observed. The vessel containing the liquid is then removed, care being taken to see that the rod is not disturbed. The microscope is then focussed on the end of the rod and the reading noted. The difference between these readings gives the height of the liquid in the capillary. These observations should be repeated. The tube is then broken at the point corresponding to the top of the meniscus and the radius found with the aid of a vernier microscope. To do this several readings of two diameters mutually at right angles are made. If the mean values of each set are equal to within about 5 per cent. the mean value can be taken as a measure of r. If the discrepancy is greater than this the tube should be rejected and another one constructed. It often saves much time if the mean diameters of the two ends of the tube are measured before commencing the experiment. If these are circular the chances will be that the rest of the tube will have a circular section. But these values must not be used in calculating γ since it is the radius at the point B, Fig. 5·12, which

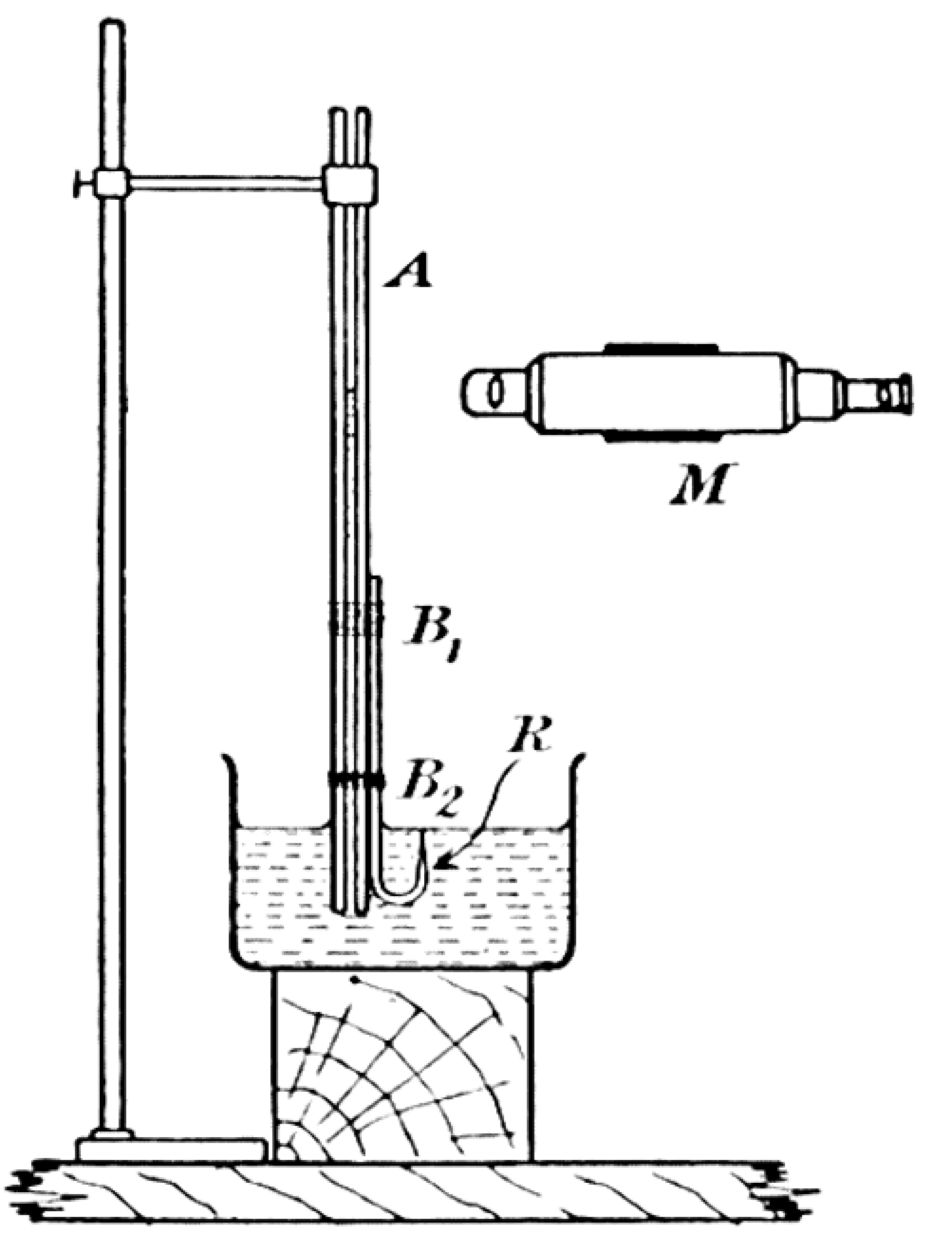


FIG. 5·14.—Measurement of Surface Tension by Rise of Liquid in a Capillary Tube.

determines the pressure change in crossing the surface of the liquid. The value of the surface tension may then be calculated from the formula already proved.

[At this point it is convenient to ask ourselves what would happen if a tube of radius r and length less than h, where h is given by $2\gamma = g\rho hr$, were dipped in a liquid of surface tension γ and density ρ. Usually, i.e. when the length of the tube is greater than h, it is the height of the liquid in the tube which adjusts itself until the equation is satisfied. When this is no longer possible, as in the problem now contemplated, the only quantity in the above equation which is a variable is r. The liquid therefore rises to the top of the tube and there forms a surface which is concave upwards and whose radius is greater than r. Its value r_1 is given by $h_1r_1 = hr$, where h_1 is the height of the liquid in the capillary.]

Note on Comparing Experimentally the Surface Tensions of Two Liquids.—If the 'rise in a capillary tube' method is adopted it is not necessary to determine the radius of the tube if the tube is arranged so that the liquid meniscus stands in turn at the same position in the tube when the heights to which the liquids rise are determined. Then

$$\gamma_1 = \tfrac{1}{2}g\rho_1h_1r, \text{ and } \gamma_2 = \tfrac{1}{2}g\rho_2h_2r.$$

$$\therefore \frac{\gamma_1}{\gamma_2} = \frac{\rho_1h_1}{\rho_2h_2},$$

and $\dfrac{\rho_1}{\rho_2}$ may be determined directly by means of Hare's apparatus.

(*b*) ***Jaeger's Method or the Method of Maximum Bubble Pressure.***—This is based on the fact that the excess pressure inside a spherical bubble of air inside a liquid is $\dfrac{2\gamma}{r}$ where r is the radius of the bubble.

The experiment consists essentially in determining the maximum pressure required to produce an air bubble at the end of a vertical capillary tube immersed in the liquid whose surface tension is being determined. A capillary tube about 0·05 cm. in diameter is constructed as in (*a*). This is placed vertically downwards in a vessel, A, Fig. 5.15 (*a*), containing the liquid whose surface tension is required. It is connected to a manometer, C, containing xylol, and also to a Woulf's bottle, D, fitted with a dropping funnel, B. Mercury (or water) is placed in B and permitted to run slowly into D. A difference of pressure between the inside and the outside of the apparatus is at once shown if the apparatus is air-tight. When the pressure in D reaches a certain value bubbles appear in A. These should be formed singly and at the rate of about one in ten

seconds. The first condition is obtained by reducing the volume of air in the apparatus so that when one bubble breaks away from the end of the capillary tube, the pressure inside the apparatus is reduced to such a value that it is less than the maximum pressure required to blow the bubble; the second condition is obtained by adjusting the rate at which liquid flows into D. The maximum height h of the manometer is recorded. If ρ is the density of the liquid in the gauge, the pressure recorded by it is $g\rho h$, where g is the acceleration due to gravity. But this pressure difference is not entirely due to the effects of surface tension, for part is attributable to the pressure due to the fact that the orifice of the capillary is at a depth d below the surface of the liquid. If σ is the density of this liquid,

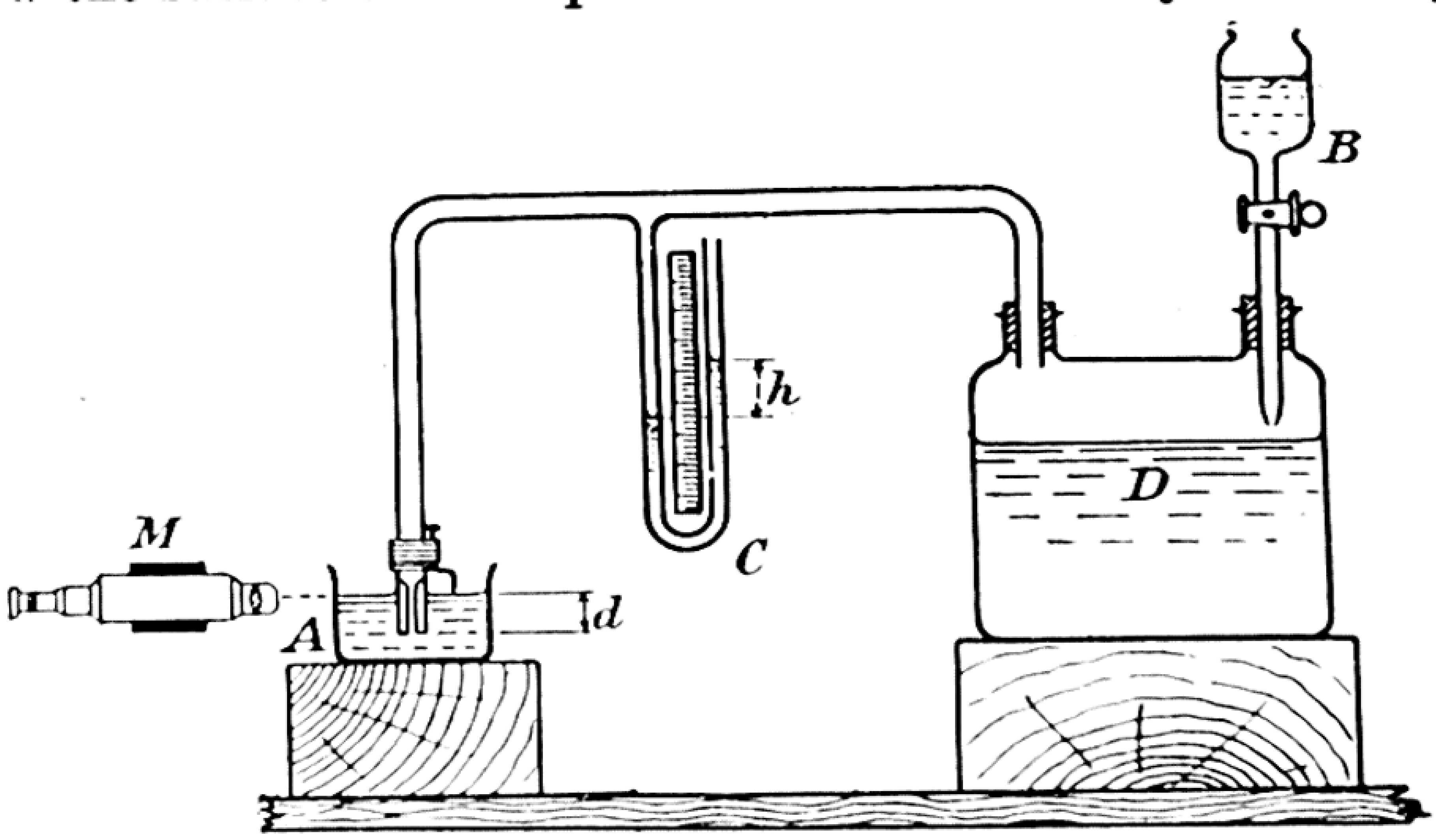


FIG. 5·15 (*a*).—Apparatus for Determining Surface Tension of a Liquid. [The microscope M is only used when the vessel A has been removed, for otherwise refraction at the curved surface of this vessel would vitiate the observations.]

this pressure amounts to $g\sigma d$, so that the pressure difference directly attributable to surface tension is $g[\rho h - \sigma d]$. We therefore have

$$\frac{2\gamma}{r} = g(\rho h - \sigma d).$$

Hence γ may be calculated when the other variables in this equation are known.

To discover the reason why the value of r used in the above equation is equal to the radius of the capillary tube at its lower end, let us suppose that the tube is uniform in diameter and that the pressure inside the apparatus is such that the centre of the hemispherical liquid surface is at C_1—cf. Fig. 5·15 (*b*). We are justified in assuming that this surface is part of a sphere if the radius of the capillary is not large, and the angle of contact between the liquid and the tube is zero. Suppose that the pressure inside the apparatus is increased so that the centre of

the surface is at C_2, the radius still being r, but that if the surface is forced down beyond this position its radius increases. When C_3 is the centre, let the radius be $(r + \delta r)$. The pressure difference across the surface is then less and the bubble grows since the pressure inside the apparatus is too great for the surface to be in equilibrium. Thus a bubble of air escapes, and the liquid surface will lie entirely above C_2, if the removal of one bubble is sufficient to reduce the pressure inside the apparatus below the maximum pressure necessary to cause a bubble to escape from the tube. If not, several bubbles will escape.

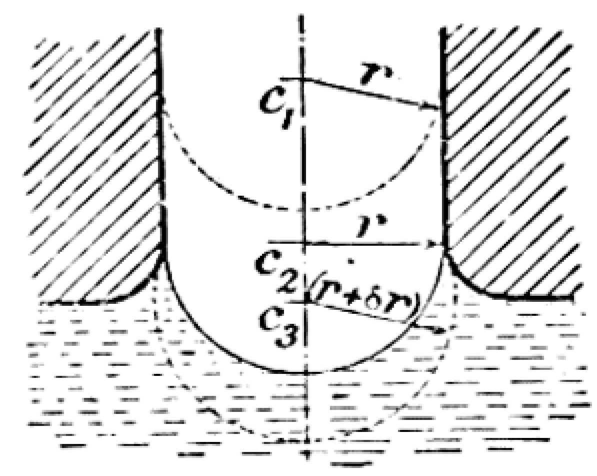


FIG. 5·15 (*b*).—Formation of a Bubble at the end of a Capillary Tube (greatly enlarged).

The great advantages of this method are that it may be applied to determine the surface tension of a molten metal, or to investigate how the surface tension of a liquid varies with temperature, or how that of a solution varies with the concentration of the dissolved substance. The method is particularly suited for such determinations as the two last, since it is not necessary to know the radius of the capillary tube. Also, since a new surface is continually being formed in the liquid the effects of contamination are reduced to a minimum, and finally the radius r can be determined before observations are made [cf. method (*a*)].

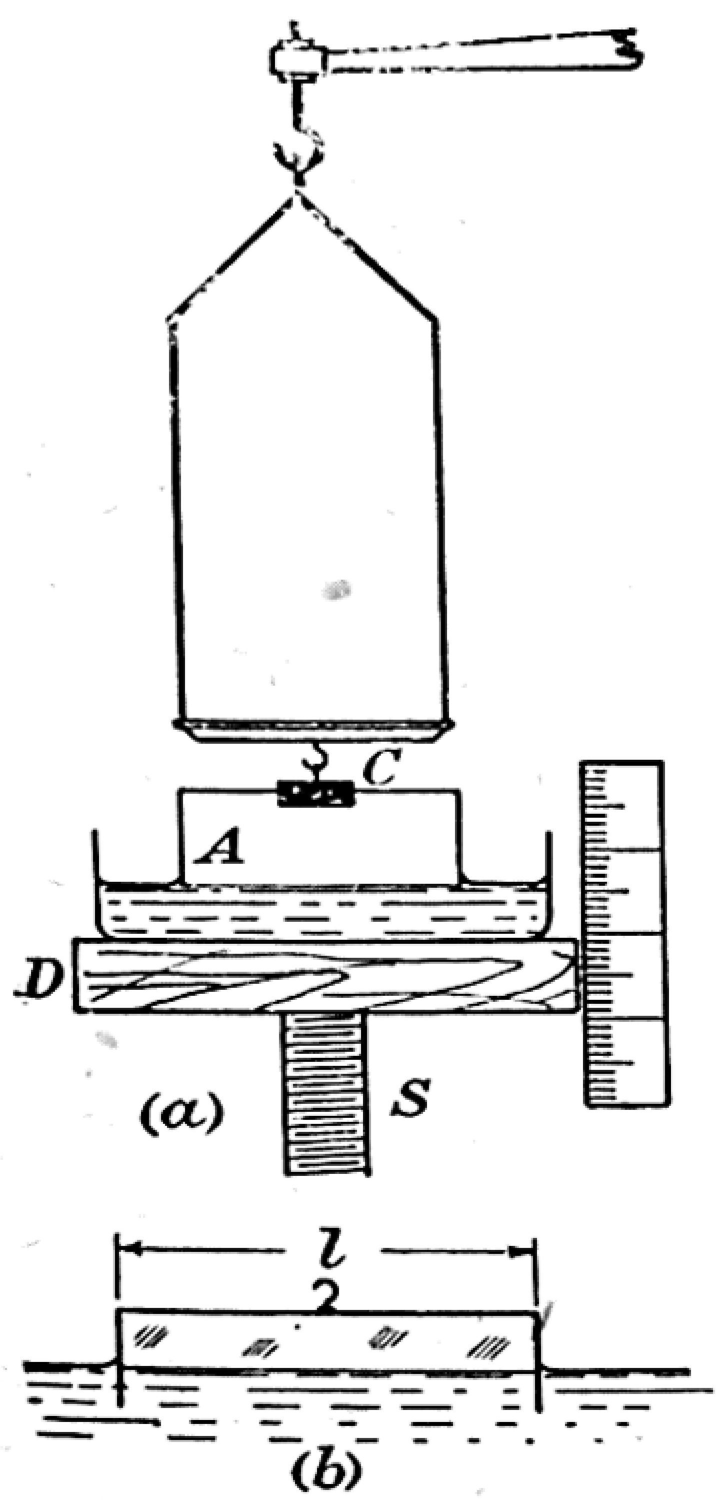


FIG. 5·16.—Surface Tension by ordinary Balance Methods.

Unfortunately certain difficulties arise when an absolute determination of the surface tension of a liquid is being made by this method. One is seldom quite sure whether or not the size of the bubbles, when the excess pressure inside the bubble is a maximum, is controlled by the internal or the external radius of the tube. If these radii differ considerably and the surface tension is known at least approximately, simple substitution of these values in the appropriate equation reveals the correct one.

In addition, although for many years it has been maintained that the method gives results which are independent of the angle of contact of the liquid with the material of the tube, PORTER has recently shown, at least for angles of contact greater than

$\pi/2$, that when the external radius is the determining one, the calculation does not involve the angle of contact, but that it is quite otherwise when the excess pressure is determined by the internal radius. Porter also remarks that it is a matter of some surprise that the belief that the results were always independent of the angle of contact should ever have gained credence, although that belief is generally held.

(c) ***Ordinary Balance Method.***—The surface tension of a liquid which wets glass may be determined as follows. A glass plate, A, Fig. 5·16 (*a*), (a microscope slide) is supported by means of a metal clip, C, from below the pan of a balance—the lower edge of the slide is made horizontal. The vessel, D, containing the liquid is placed on a small table below the slide. The table may be raised by means of a screw, S. The balance is equilibrated and left free to swing. The adjustable table is then screwed up till the liquid ***just*** touches the lower edge of the plate. This is shown by a sharp jerk of the pointer as the microscope slide is pulled down by surface tension. Masses are then added to the other pan of the balance until the slide is withdrawn from the liquid. Since the lower edge of the slide had been in the general level of the liquid surface there is no correction for buoyancy. If l is the length and t the thickness of the slide at its lower edge, the force due to surface tension acting on it is $2(l + t)\gamma$. This is equal to mg, where m is the mass added to the pan to restore equilibrium. Hence γ may be determined.

An alternative method is as follows. Having screwed up the adjustable table till the pointer jerks, observe the position of the table (suitable scales may be arranged as on a spherometer). Instead of restoring equilibrium as above, the table is screwed up through a distance h until the pointer is back at zero. Then the buoyancy force just balances the force due to surface tension, and if the vessel containing the liquid has a large surface area, so that h will be also the depth of immersion of the slide, then

$$2(l+t)\gamma = lth\rho g,$$

where ρ is the density of the liquid.

It must be noted that this method only yields accurate results if the liquid completely fills the containing vessel so that the surface of the liquid may be cleaned with the aid of waxed pieces of glass, as described on p. 118.

Soap Solutions.[1]—The plate method described above may

[1] Prof. Boys recommends the following soap solution. To a litre of distilled water contained in a well-stoppered bottle add 25 gm. of sodium oleate, and let it stand for 24 hours. Then add about 300 cm.^{3} of glycerol, shake well, and allow to stand for a week. By means of a siphon remove the clear

easily be adapted to determine the surface tension of a soap solution. A glass or wire frame, as shown in Fig. 5·16 (*b*), is made and is supported from below one pan of a balance, and arranged that when the balance is equilibrated, the horizontal portion of the frame is about 0·5 cm. above the general surface of the liquid. The frame is then immersed completely and extra masses, *m*, added to the right-hand balance pan until the frame is in the same relative position as before. If *l* is the length of the horizontal portion, the weight of the film being negligible,

$$2\gamma l = mg.$$

[This method may be used for liquids such as water, the horizontal portion of the frame then being nearer to the general surface of the liquid.]

Drops and their Formation.—Suppose that a glass tube about 2 mm. in diameter has been connected to a wide tube by means of rubber tubing and a narrow capillary glass tube, and the whole filled with a liquid—say, water. The capillary tube is merely to control almost entirely the rate at which the liquid escapes when the apparatus is held in a vertical position with the narrow tube pointing downwards. If the water leaving the tube is carefully watched it will be seen to assume, in turn, shapes whose outlines are shown in Fig. 5·17 (*a*) and (*b*). As the drop continues to grow a waist is formed—the drop is then about to break away —cf. Fig. 5·17 (*c*). When this occurs the water comprising the neck will form a small sphere following the larger drop. It is known as ***Plateau's spherule*** —cf. Fig. 5·17 (*c*).

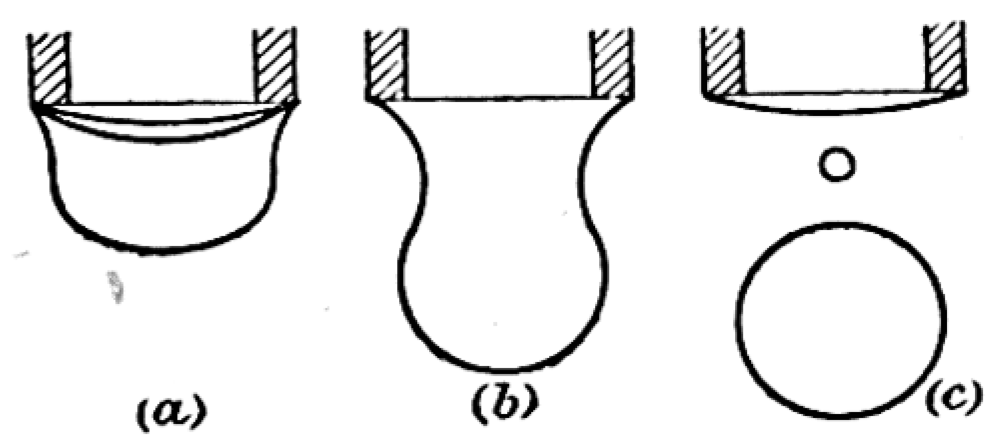


FIG. 5·17.—Drops and their Formation. (*a*) Early Stages in the formation of a Drop. (*b*) Drop showing the formation of a Waist. (*c*) A large Drop and Plateau's Spherule.

To observe more easily the formation of a drop of liquid it is necessary to diminish the effective pull of gravity on the drop. This was done in a very striking way by DARLING. At temperatures above 80° C. the density of aniline is less than that of water at the same temperature, whereas the reverse is true at lower temperatures. Moreover, aniline and water are immiscible. Suppose, therefore, that a large tall beaker is nearly filled with water and a quantity of aniline (about 100 cm.^{3}) added. This collects at the

liquid, leaving the scum behind. Add two or three drops of liquid ammonia to the solution and store in a dark cupboard. The solution must not be warmed or filtered.

bottom of the beaker. A bunsen burner is then placed below the beaker: when the aniline assumes a temperature of about 80° C. it ascends to the top of the water and collects there in the form of a pendant drop. The rate of supply of heat is diminished and the aniline cools: a large drop about 3 cm. in diameter begins to form. The drop then has a distinct neck which gradually becomes more thin. Finally, two constrictions are formed, and a large drop of aniline, followed by Plateau's spherule, falls to the bottom of the beaker. The process is then repeated.

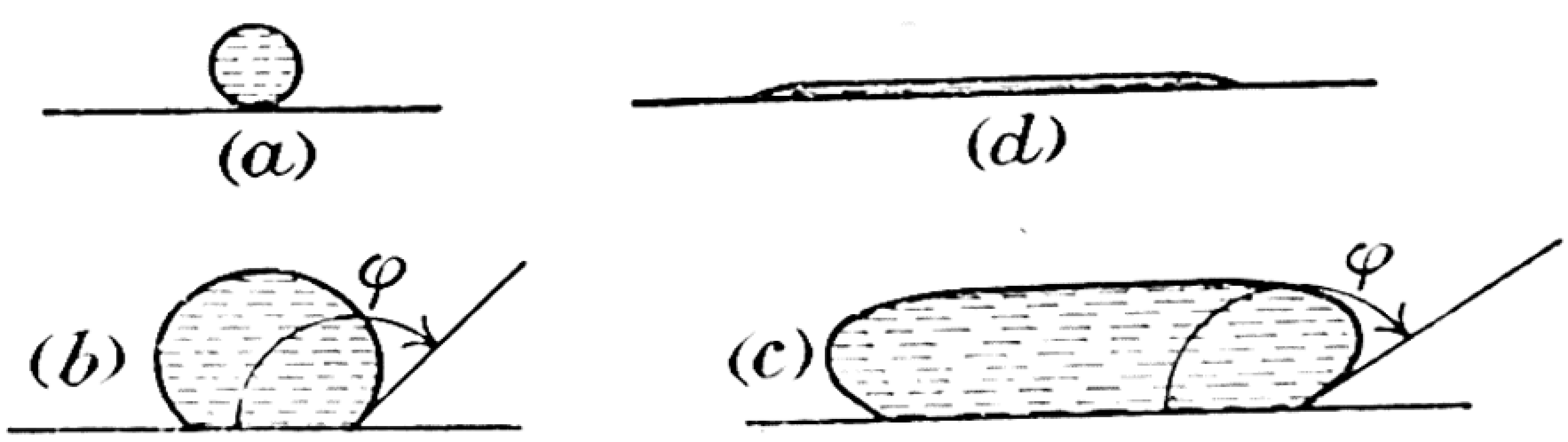


FIG. 5·18.—Wetting of a Surface by a Liquid.

Drops of liquid not wetting the surface with which they are in contact—say, mercury—are, when small, truly spherical—cf. Fig. 5·18 (*a*). As the drop grows, or if several small ones coalesce, it loses its sphericity—cf. Fig. 5·18 (*b*)—while a very large drop is perfectly flat except near its edges, i.e. it assumes the shape shown in Fig. 5·18 (*c*).

If the liquid wets the surface, drops do not form, and the liquid spreads itself over the surface—Fig. 5·18 (*d*).

A Liquid Drop between Plates.—When two clean pieces of plate glass are placed face to face no difficulty is experienced in separating them, but a considerable force is necessary to pull them away when a small drop of water, for example, is placed between them and the plates are very close together. If d is the distance apart of the plates and the area wetted is large, then we may regard any small element of the liquid surface as part of a cylindrical one with a radius of curvature $0{\cdot}5d$. Consequently the pressure in the water is less than that outside by an amount $\frac{2\gamma}{d}$. If A is the area of each plate which is wetted, the total force pulling the plates together is $\frac{2\gamma}{d}.\mathrm{A}$.

On the other hand, when a small drop of mercury is placed between two plane surfaces, considerable force must be applied to the plates in order to flatten the drop to any extent. If ϕ is the angle of contact between mercury and glass, the radius of curva-

ture, R, of the mercury surface normal to the plane of the plates is given by

$$\frac{\frac{d}{2}}{R} = -\cos\phi.$$

∴ Pressure in the mercury is greater than atmospheric by

$$\frac{\gamma}{R} = -\frac{2\gamma\cos\phi}{d}.$$

Since $\frac{\pi}{2} < \phi < \pi$, the above expression is positive. The force to be applied is therefore $\frac{2\gamma\, A\cos(\pi - \phi)}{d}$.

Surface Films on Water.—Many pure substances of a fatty nature, when placed on a clean surface of water, spread themselves out to form an exceedingly thin surface layer. It can be shown that the thinnest film which can be formed on water is one molecule in thickness, each molecule of the oil being in direct contact with the surface of the water.

Some extremely interesting conclusions have been drawn from the study of these layers, for their simple structure makes them peculiarly suitable for investigating the properties of the molecules themselves. It is found that these films can exist in three forms, corresponding to the solid, liquid and gaseous states of matter. In the 'gaseous' state of the films, the molecules move about individually and separately in the surface, exerting an outward spreading force on the boundary of the surface, in much the same manner as a gas exerts a pressure on the walls of the vessel containing it, or a dissolved substance exerts an osmotic pressure on a semi-permeable membrane. In the 'solid' and 'liquid' states of the film the molecules adhere into compact, coherent masses, in which they are often just as closely packed as in solids or liquids in bulk.

In these coherent films the cross-sectional area of the individual molecules has been measured by measuring the area of the film composed of a known number of molecules, as calculated from the mass of the film, i.e. the mass of the drop of substance placed on the water surface, and the mass of one of its molecules. The results of such measurements show that the molecules actually have the shapes which have been indicated for about three-quarters of a century by the structural formulæ of organic chemistry. It is found, for instance, that the molecule of stearic acid, $C_{17}H_{35}COOH$, is just about five times as long as it is thick; that the end group (COOH), under certain circumstances, is slightly thicker than the rest of the molecule; and that usually the molecules pack into a coherent layer,

standing nearly vertical with the COOH groups directed towards the water. Many other coherent films, though not all, have the same vertical disposition of the molecules. In the 'gaseous' films, when the molecules do not cohere, they lie flat upon the surface of the water.

Viscosity.—Whenever relative motion exists between the different layers into which we may imagine a liquid is divided, forces are called into play tending to retard the more rapidly moving layers and to accelerate those which are moving more slowly. Similar forces, although much smaller, arise when a gas moves in the same way. To obtain a more definite idea of these forces let us consider Fig. 5·19 (*a*). In this xOy represents the boundary between a fluid and a solid over which the former is flowing. At this boundary it will be assumed that the fluid is at rest, and that all the molecules in a plane parallel to xOy have a resultant motion

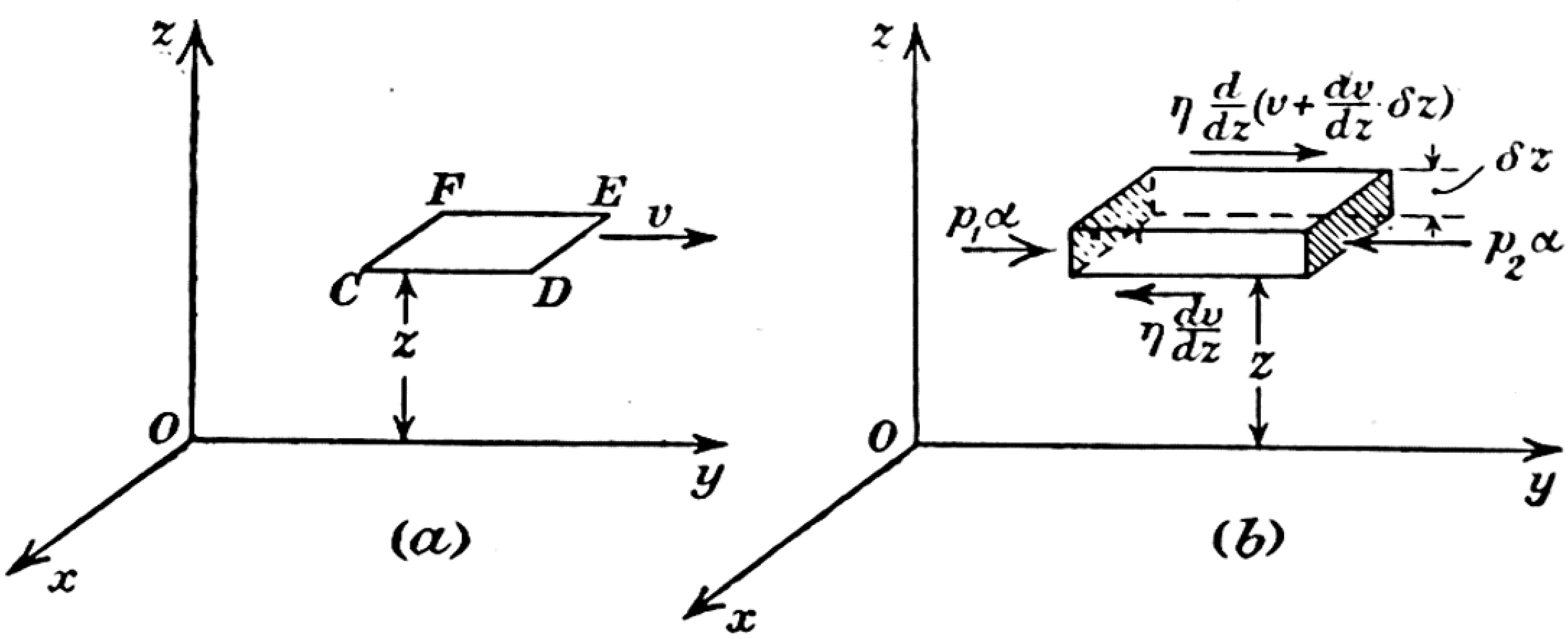


FIG. 5·19.—Coefficient of Viscosity.

(mass-velocity of the fluid) which is parallel to the above reference plane and which increases with the distance of the layer from that plane. Let CDEF be an area of magnitude S at distance z from the reference plane. Then the molecules immediately above this plane tend to accelerate the molecules in it, while the molecules in the layer immediately below tend to retard them. In this way each stratum of fluid will exert on the one next to it a tangential traction, opposing the relative motion between the two layers. If F is the magnitude of this tangential force, the force acting on unit area of CE is $\frac{F}{S}$: this is the tangential stress due to viscosity in the fluid. We assume that the magnitude of this stress is directly proportional to the difference in velocity between the layers immediately above and below the plane considered, divided by their distance apart. This latter quantity is termed the ***velocity-***

gradient in the fluid. It is denoted by $\frac{dv}{dz}$, where v is the mass-velocity of the fluid at a height z above the reference plane. We may therefore write

$$\frac{F}{S} = \eta\frac{dv}{dz},$$

where η is a constant called the ***coefficient of viscosity*** of the fluid. It depends upon the nature of the liquid and its temperature. [Notice the similarity between this definition and those of diffusion and of thermal conductivity.]

The value of η in C.G.S. units is expressed in dynes per square centimetre per unit velocity gradient, i.e. gm. cm.$^{-1}$ sec.$^{-1}$ This unit is often called the '***poise***' in honour of POISEUILLE.

The above equation cannot be verified directly, but calculations based on it are in strict accord with experiment so that we do not hesitate to accept the above equation as a complete statement of the laws of viscosity.

To determine the relation between the viscous forces in a fluid and the pressure differences in it, consider the volume of fluid lying between planes at heights z and $z + \delta z$ above xOy—cf. Fig. 5·20 (*b*). Let the area of the faces parallel to xOy be unity, and let α be the cross-sectional area of the element in a direction normal to Oy.

Then the forces due to viscosity acting on the lower and upper faces are

$$\eta\frac{dv}{dz} \text{ and } \eta\frac{d}{dz}\left(v + \frac{dv}{dz} \cdot \delta z\right),$$

their lines of action being parallel to yO and to Oy respectively. Let p_1 and p_2 be the pressures at points on the two ends of the prism, $p_1 > p_2$; the forces are $p_1\alpha$ and $p_2\alpha$ as indicated. Since the fluid is moving without acceleration, the total force on the element considered must be zero. Hence

$$\eta\frac{d}{dz}\left(v + \frac{dv}{dz} \cdot \delta z\right) - \eta\frac{dv}{dz} + p_1\alpha - p_2\alpha = 0.$$

$$\therefore\ \eta\frac{d^2v}{dz^2} \cdot \delta z = (-p_1 + p_2)\alpha.$$

Experimental Determination of Viscosity.—***Method i:*** To determine the viscosity of water the apparatus shown in Fig. 5·20 may be used. It consists of a tall metal cylinder furnished with an overflow pipe DC. A capillary tube of known length, l, and radius, r, is placed in a horizontal position and connected to the cylinder. Water enters along the inlet tube as shown, any excess being carried away along DC. Attached to the exit end of the capillary is a glass tube bent in the manner indicated. The pressure difference between the ends of the capillary is proportional to the

vertical distance between the levels A and B. This may be determined with the aid of the scale in mm. and a U-tube filled with water and placed as shown so that the levels at A and C are the same. If the water is allowed to flow along the tube, as each drop breaks away from E the water level at B changes—an effect due to the changes in pressure at E as the drops alter in shape. This disturbing factor may be avoided if a small clean glass rod is placed in contact with the liquid. The liquid then leaves the tube in a trickle and the level at B is constant.

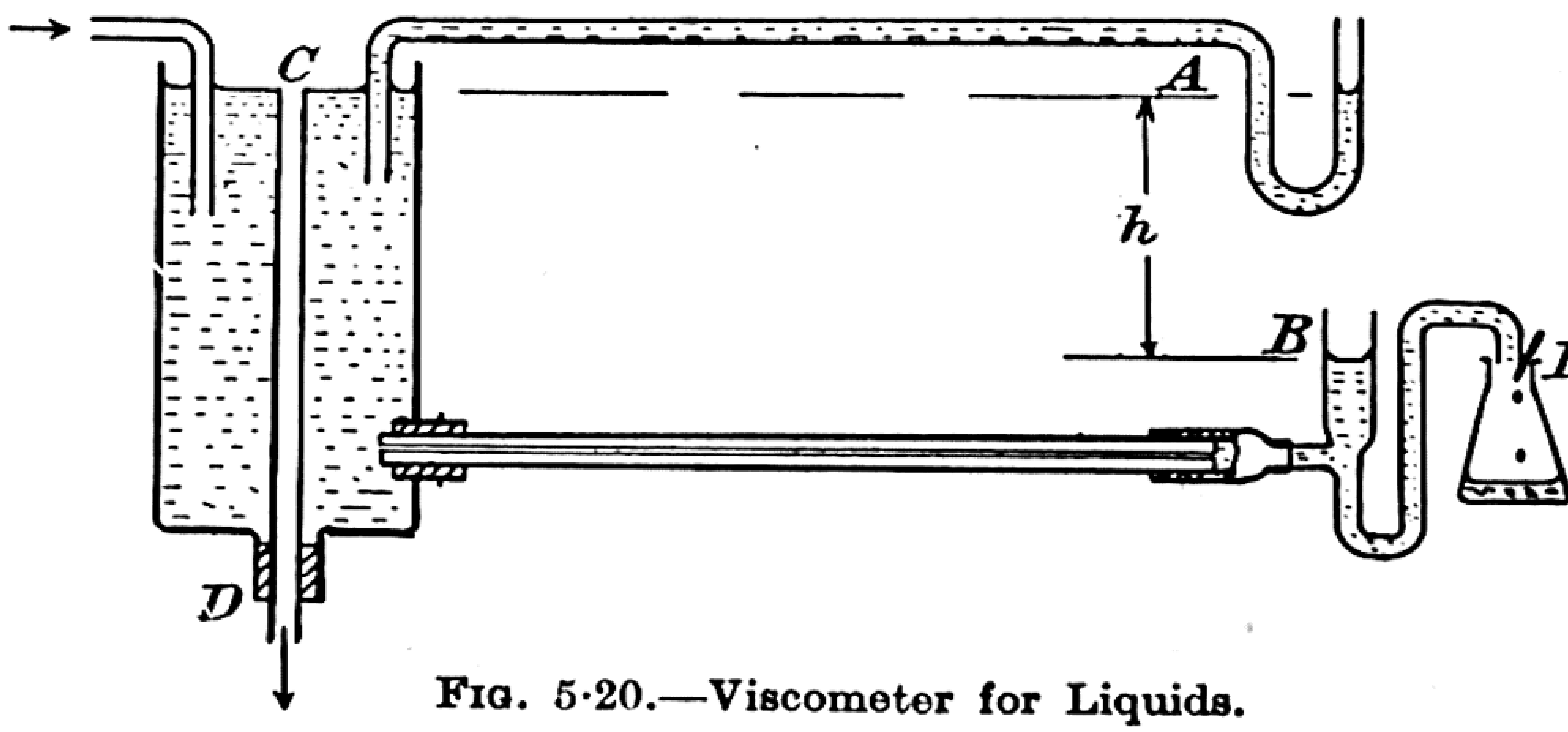


FIG. 5·20.—Viscometer for Liquids.

It may be shown, by reasoning beyond the scope of this book, that v, the volume of liquid emerging in t seconds from the tube, is given by

$$v = \frac{\pi r^4 pt}{8l\eta},$$

where η is the coefficient of viscosity, p the pressure difference between the ends of the capillary—it is $g\rho h$, where $h = \text{AB}$, ρ is the density of the water, and g is the acceleration due to gravity. It must be pointed out that the formula is true only for narrow tubes in which the velocity of the liquid is not so great that the flow becomes turbulent.

Stokes' Law.—When a sphere falls vertically downwards through a viscous medium, the layers of liquid adjacent to the sphere tend to move with a velocity equal to that of the sphere. At a great distance from the sphere the liquid is at rest. Consequently there must be relative motion between the different layers of the liquid and the motion of the sphere will depend on the viscosity of the medium. If the sphere is small it is found that it soon acquires a constant velocity, i.e. the pull due to gravity on the sphere is balanced by the upthrust of the liquid on it and the force arising from its motion through the viscous medium.

This vertical force, F, will depend on η, the viscosity of the medium,

a, the radius of the sphere, and v the constant or terminal velocity acquired by the sphere. Thus

$$F = \kappa a^{\alpha} \eta^{\beta} v^{\gamma}$$

where κ is a constant, and α, β, and γ are the appropriate dimensional coefficients. In addition to the dimensions of F, a, g, and v, which are already known, we require those of η. Now

$$\frac{[\text{force}]}{[\text{area}]} = [\eta]\left[\frac{dv}{dx}\right].$$

Hence
$$\frac{[\text{MLT}^{-2}]}{[\text{L}^2]} = [\eta]\frac{[\text{LT}^{-1}]}{[\text{L}]},$$

so that $[\eta] = [\text{M}][\text{L}]^{-1}[\text{T}]^{-1}$.
We therefore have

$$[\text{MLT}^{-2}] = [\text{L}]^{\alpha}\,[\text{ML}^{-1}\text{T}^{-1}]^{\beta}\,[\text{LT}^{-1}]^{\gamma}.$$

Equating like exponents, we have

$$\beta = 1,\ \alpha - \beta + \gamma = 1,\ \beta + \gamma = 2.$$
$$\therefore\ \gamma = 1,\ \alpha = 1.$$
$$\therefore\ F = \kappa a \eta v,$$

and it can be shown that $\kappa = 6\pi$, i.e. $F = 6\pi a \eta v$.

This expression was first obtained by STOKES, and is known as Stokes' law for the force acting on a sphere falling under gravity through a viscous medium.

The Viscosity of Oils.—Suppose that ρ is the density of the material of the sphere, σ that of the liquid. Since

Weight of sphere = upthrust due to liquid displaced + force due to the motion of the sphere, we have,

$$\tfrac{4}{3}\pi a^3 \rho g = \tfrac{4}{3}\pi a^3 \sigma g + 6\pi a \eta v.$$
$$\therefore\ \eta = \tfrac{2}{9}a^2 g\frac{(\rho - \sigma)}{v}.$$

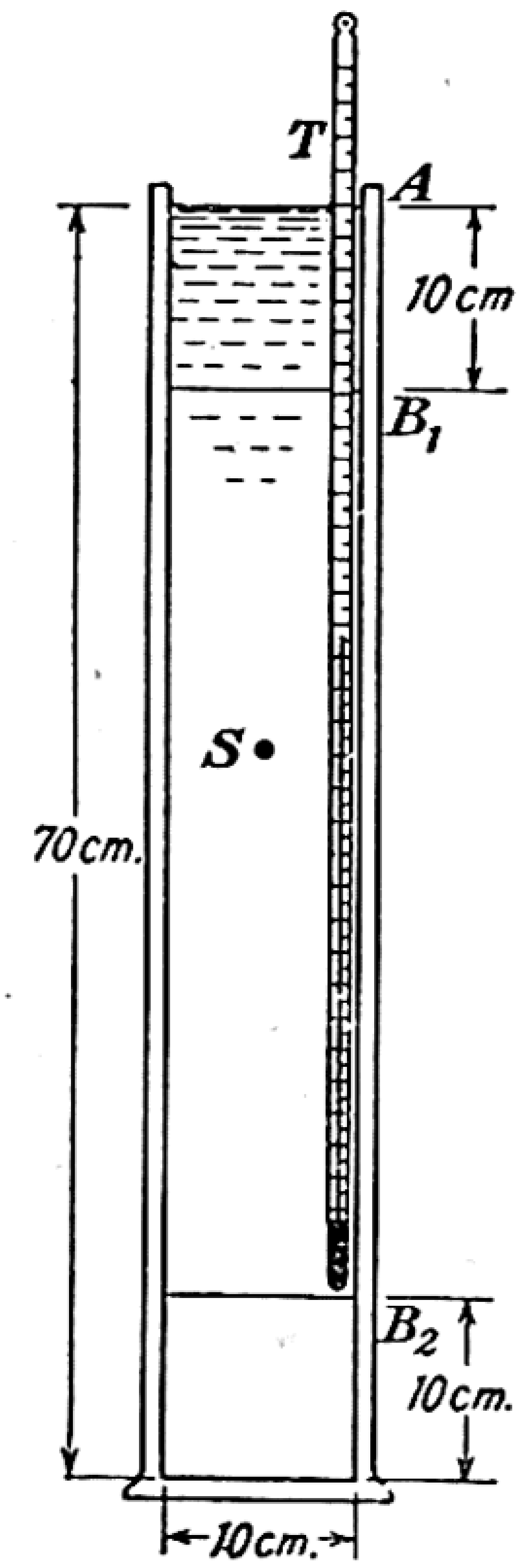


FIG. 5·21.—Viscosity of Oils—Stokes' Method.

The above expression shows that if the velocity of fall of a sphere through a viscous medium can be measured, we have a means of determining the coefficient of viscosity of the medium.

Let us suppose that glycerol is the liquid whose viscosity is to be determined. This is placed in a glass cylinder, A, Fig. 5·21, about 70 cm. long and 10 cm. wide. Spheres of known diameter are dropped into the liquid and the terminal velocity for each sphere is deduced from observations on the time required for the sphere to travel between two fiducial marks. Now the liquid is limited by the walls of the vessel and has a finite depth. The conditions stipulated by the above theory are therefore not fulfilled. It may be shown, however, that if the sphere falls between two fiducial marks B_1 and B_2 (10 cm. from the top and bottom of the liquid respectively), then the motion is uniform. Further, if the diameter of the sphere does not exceed 0·2 cm. and a vessel 10 cm. wide is used, no correction is necessary for the effect of the walls of the vessel. If λ is the distance between the fiducial marks, and t the time of transit,

$$\eta = \tfrac{2}{9}a^2 g\left(\frac{\rho - \sigma}{\lambda}\right)t,$$

so that a^2t is constant for a given liquid at a constant temperature. If therefore a^2 is plotted against $1/t$ a straight line should be obtained if the conditions of the theory have been satisfactorily fulfilled. From the slope of the line η may be deduced.

Since the viscosity of an oil changes very rapidly with temperature it is advisable to measure and record the latter to within 0·1° C.—T is a thermometer—and, having measured the diameters of all the spheres to be used in some definite experiment, to carry out the fall experiments one after the other as quickly as possible. The spheres must fall centrally down the tube A, so that their fall shall not be affected by the walls of the tube.

EXAMPLES V

1.—How would you proceed to determine the osmotic pressure of a solution? Give some account of plasmolysis and isotonic solutions.

2.—Describe the apparatus used for preparing colloids by means of hot dialysis.

3.—A liquid whose density is 0·83 gm. cm.$^{-3}$ rises to a height of 8·92 cm. in a tube whose diameter is 0·0168 cm. What is the surface tension of the liquid?

4.—Describe a method of determining the viscosity of an oil. In an experiment with the cup and ball viscometer the time to break away for an oil of known viscosity 6·3 gm. cm.$^{-1}$ sec.$^{-1}$ was 60·7 sec. What is the viscosity of an oil when the time is 26·2 sec.?

5.—Define the terms *surface tension* and *surface energy*. Give the theory of one method of determining the surface tension of a liquid whose angle of contact with glass is zero. How would you demonstrate the existence of surface energy in a liquid film?

6.—What determines whether a liquid will rise or fall in a capillary tube placed with one end below the surface of a liquid? How may the surface tension of a molten metal be determined?

7.—Explain the terms *osmosis* and *osmotic pressure*. Upon what factors does the osmotic pressure of a solution depend?

8.—A glass microscope slide, 10 cm. long and 1 mm. thick, is suspended from one arm of a balance so that its lower edge is horizontal and its plane vertical. The balance is left free and equilibrated. A vessel containing alcohol is placed below the slide and then raised until the alcohol just touches the lower edge of the slide. If a mass of 0·63 grams must be placed in the opposite pan of the balance to restore equilibrium, calculate the surface tension of alcohol.

9.—Define the coefficient of viscosity and describe how you would proceed to compare the viscosities of two liquids—say alcohol and water—at room temperature.

10.—Describe and explain how the surface tension of a liquid may be measured by forcing bubbles of air through it. Discuss whether the result obtained in this way should be the same as that given by the capillary tube method.

11.—Two vertical plates, distance d apart, are immersed in a liquid whose angle of contact with the plates is zero, and whose surface tension is γ. Calculate the height to which the liquid will rise at a point some distance from the edges of the plates.

12.—Discuss the shape of a liquid surface in the space between two vertical plates inclined at a small angle to one another.

13.—Describe and explain what happens when minute camphor particles are scattered on a clean water surface. Why does immersing one's finger in the water modify the effect? A spherical soap bubble of radius 2 cm. is blown in an atmosphere whose pressure is 10^6 dyne. cm.$^{-2}$. If the surface tension of the liquid composing the film is 60 dyne. cm.$^{-1}$, to what pressure must the surrounding atmosphere be brought in order exactly to double the radius of the bubble? Assume no temperature change and no diffusion through the bubble. (N.H.S.C. '29.)

14.—State and give the theory of a method of determining the surface tension of mercury, in which measurement of the angle of contact between mercury and glass can be avoided. (L. '23.)

15.—Define *surface tension* and *angle of contact*. If the surface tension of a liquid having a density of 0·82 gm. cm.$^{-3}$, is 28·3 dyne. cm.$^{-1}$, calculate the height to which the liquid will rise in a glass capillary tube of 0·5 mm. diameter dipped into it, the angle of contact between the liquid and glass being 30°.

16.—A U-tube with vertical limbs is half-filled with liquid. If the diameters of the two limbs are 1 cm. and 0·1 cm. respectively, calculate the difference in height of the liquid in the two limbs if the density of the liquid is 1·27 gm. cm.$^{-3}$ and its surface tension is 45 dyne. cm.$^{-1}$ Assume the angle of contact to be zero.

17.—A capillary tube 0·15 mm. in diameter has its lower end immersed in a liquid whose surface tension is 54 dyne. cm.$^{-1}$ and whose density is 0·86 gm. cm.$^{-3}$ Calculate the height to which the liquid rises, the angle of contact being 28°. Establish the formula used.

CHAPTER VI

ELASTICITY

Strain and Stress.—A system of forces acting on a body may sometimes be such that although there is no motion of the body as a whole yet there may be a relative displacement of its constituent particles causing a change of form or a change in the dimensions of the body. Such a body is said to be ***strained.*** When a body is strained forces are called into play tending to resist the relative displacement of the component particles: the body is then said to be in a ***state of stress.*** There are three types of simple strain and simple stress: (*a*) tensile strain and tensile stress, (*b*) compressive strain and compressive stress, and (*c*) strain and stress caused by shear.

Tensile Strain and Stress.—In Fig. 6·1 (*a*), AB represents a uniform bar of initial length L. When stretching forces FF act upon AB its length increases by an amount l when equilibrium is attained, i.e. the internal forces in the body have reached such a magnitude that a further displacement of the component particles of the body is prevented. The ratio $\frac{l}{L}$ is called the ***tensile strain*** of the body and since both l and L are lengths, this strain, like every other strain, is measured by a mere number.

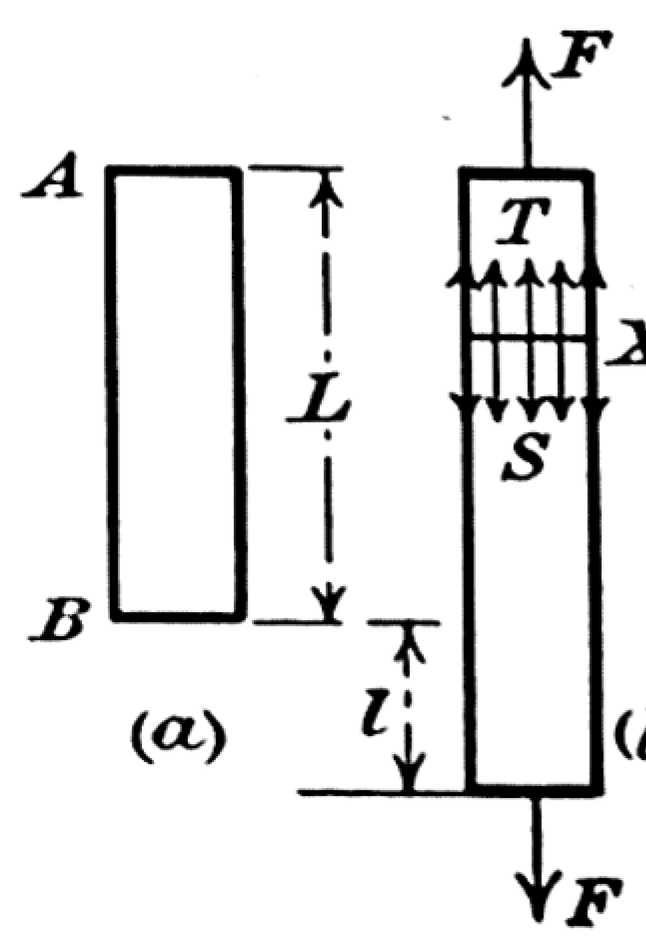


FIG. 6·1.—Tensile Strain and Stress.

Taking any arbitrary and imaginary section in the bar normal to its length as at X, Fig. 6·1 (*b*), the internal forces across this section are such that the forces S just balance the force F at A, while the forces T just balance F at B. These internal forces resist the efforts of the forces FF to break the bar: they constitute a ***tensile stress.*** Since these internal forces are distributed over an area the ***stress*** is measured by the force per unit area, so that stresses are expressed in the absolute systems of units either as dyne. cm.$^{-2}$, or as poundal.ft.$^{-2}$ Since the resultant of the

internal forces S or T at X is F, if P is the stress and s the area of cross-section at X, we have $P = \frac{F}{s}$.

Compressive Strain and Stress.—If the forces FF acting on the above body were reversed the length would decrease by an amount l, the body would be subject to a ***compressive strain*** of amount $\frac{l}{L}$, and the stress due to compression would be $\frac{F}{s}$.

Shear Stress and Strain.—A shear stress exists between two parts of a body in contact when each part exerts an equal and opposite force laterally on the other part and in a direction tangential to the surface of contact separating the two parts. Thus, suppose a rivet holds two plates together which sustain a pull F,F, across the section AB, Fig. 6·2. Under these conditions the lower portion of the rivet exerts a force parallel to AB on the upper portion, preventing it from moving to the left : similarly, the upper part exerts a force on the lower. The rivet is said to be

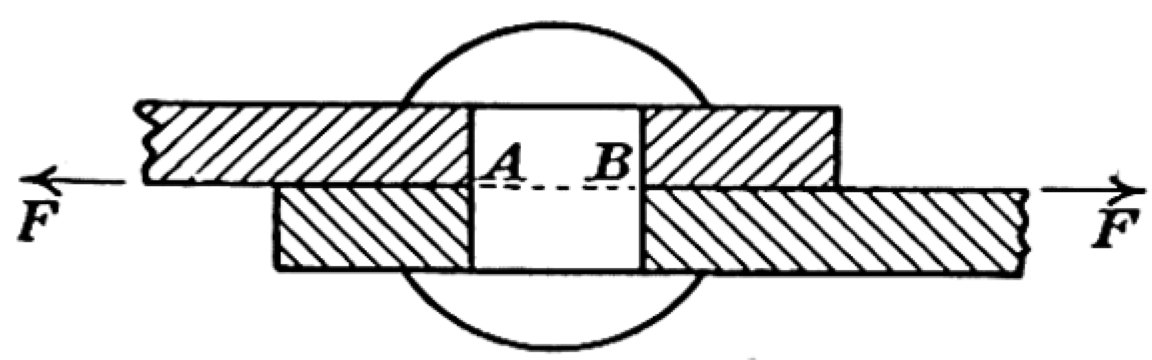


Fig. 6·2.—Shear Stress and Strain.

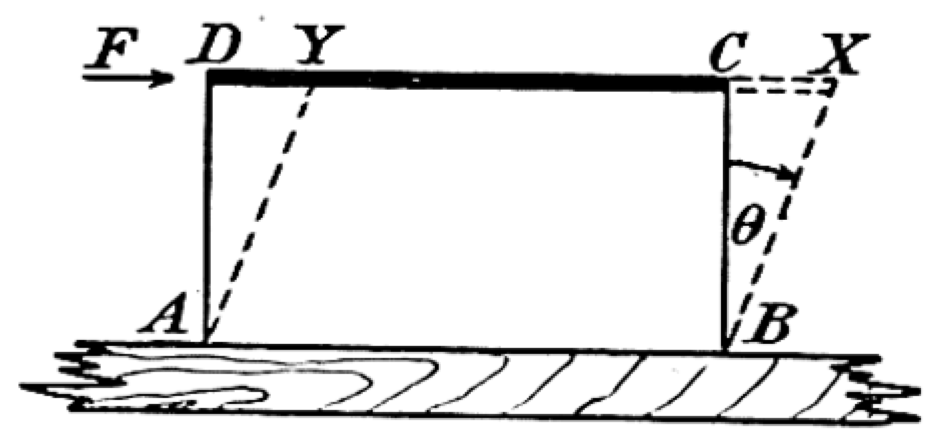


Fig. 6·3.—Shearing Strain.

in a ***state of shear*** across the plane AB, and if s is the area of the section AB, the ***shear stress*** is defined as F/s.

To discover how the strain is measured when a state of shear exists, let us consider ABCD, Fig. 6·3, the cross-section of a block of india-rubber glued to a table along that face of which AB is the trace. Imagine that a piece of sheet brass glued to the upper surface is urged forward by a force F parallel to AB. When equilibrium is reached let the plate be in position XY, i.e. the plate will have suffered a displacement CX with respect to the lower face. The block is now said to be sheared, the amount of the shearing strain being specified by the ratio $\frac{CX}{BC}$, i.e. $\tan\theta$, where θ is the $\widehat{CBX}$. It will be seen that the shearing strain is the ratio of the relative lateral displacement CX of two horizontal layers at distance BC apart to that distance, i.e. it is equal to the numerical value of the relative lateral displacement of two horizontal layers at unit distance apart.

If s is the area of the upper face the shearing stress is $\frac{F}{s}$.

It is important to note the following distinction between strain due to stretching [or compressing] forces and that due to shearing forces, for in the first instance both the volume and shape of the body may alter, whereas in the second it is the shape alone which changes, the volume remaining constant. A particular instance in which a change in volume but no change in shape occurs is when a cube of material which is isotropic, i.e. has properties the same in all directions, is subjected to a uniform pressure.

Complimentary Stresses due to Shear.—Theorem: ***A shear stress in a given direction cannot exist without an equal shear stress existing at right angles to it.*** To prove this, let us consider the rectangular body of sides, a, b, and c, shown in Fig. 6·4. Let F_1, F_1, be the forces tending to displace the upper face with respect to the lower. The area of each of these faces is ab, so that the shear stress is F/ab. Let F_2,F_2 be shearing forces at right-angles to the above. Then the corresponding stress is F_2/ac. For equilibrium, the moment of all the forces about any point in their plane—say A—must be zero, i.e. $F_1 . c = F_2 . b$. Dividing throughout by abc, we have

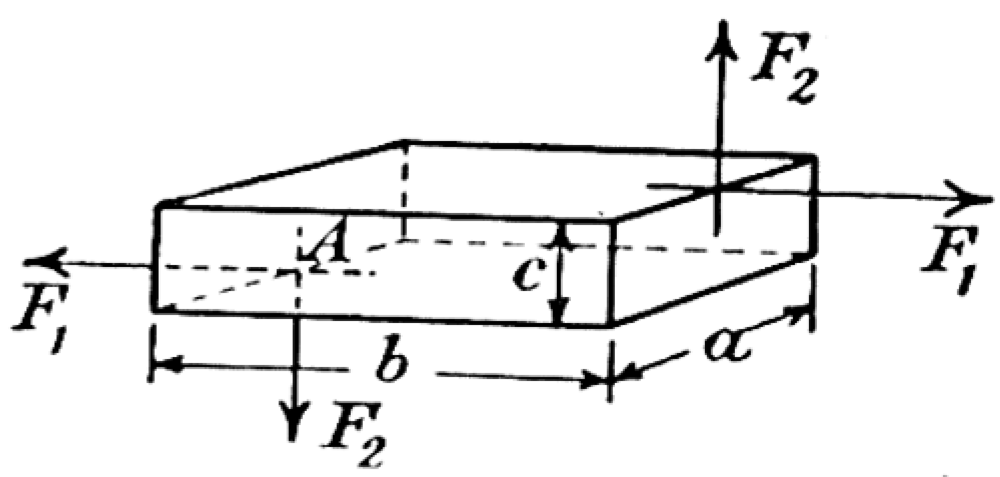


Fig. 6·4.—Shear Stress and Strain.

$$\frac{F_1}{ab} = \frac{F_2}{ac},$$

i.e. the stresses are equal.

Elasticity.—When the forces acting on a strained material are removed the body may assume its original form and dimensions. Such a body is said to possess ***elasticity.*** Thus a piece of india-rubber is very ***elastic,*** while lead and putty are almost ***non-elastic.***

Hooke's law and the Limit of Perfect Elasticity.—We have just defined the terms elastic and non-elastic as if they applied to two essentially different classes of substance. Actually, all bodies are elastic to a certain degree, depending on the magnitude of the applied load. Thus, if lead is subjected to small stretching forces it recovers its original form and size when the forces are removed —i.e. the lead then behaves as an elastic body. On the other hand, lead is non-elastic when the forces are not small and it is said to acquire a ***permanent set.*** The limit of stress within which the strain in a given material completely disappears when the stress is removed is called the ***elastic limit,*** or ***limit of perfect elasticity.*** At stresses below the elastic limit there is a linear relationship between a stress and the corresponding strain: this

fact was discovered in 1679 by HOOKE, a contemporary of BOYLE, and is known as ***Hooke's law.***

The existence of the elastic limit is very strikingly shown by the following experiment :—Two long pieces of copper wire of the same diameter are suspended from the ceiling and an electric current passed through one of them so that it just glows in a darkened room. When the wire is cool, pans are attached to each wire and loaded with equal masses which are increased by 250 or 500 gm. at a time. At first the elongation of each wire is of the same order of magnitude and if the loads are removed the wires will resume their original lengths. On increasing the loads further, a stage is soon reached when the wire which has been heated extends very rapidly and when the load is removed it is found to have acquired a permanent set. From this experiment it is clear that the elastic properties of a given material depend on its previous history. The heated wire is in an annealed condition, whereas the other wire which has been manufactured by drawing it through a die [a small hole in a steel plate—called a ***Wurtel*** plate] is said to have been ***cold worked.*** The effect of cold-working a metal by drawing it through a die, by rolling it in a mill, or by hammering it, is to increase its hardness, to lower its ductility, and to diminish its capacity for resisting mechanical shocks.

Young's Modulus and the Modulus of Rigidity.—The fundamental law relating to elasticity, discovered by HOOKE, is that the strain is proportional to the stress by which it is produced, provided that the elastic limit has not been exceeded. This relationship may be written

$$\text{stress} = k \times \text{strain}$$

where k is a constant in any given instance. This constant is called the ***modulus of elasticity*** and depends upon the nature of the material and the type of stress used to produce the strain. When the body is subject to a simple tension (or compression), the body being free to contract in a direction normal to the line of action of the stretching forces, k, i.e. the ratio $\frac{\text{stress}}{\text{strain}}$, is called ***Young's modulus.*** When the stress is due to shear the ratio $\frac{\text{stress}}{\text{strain}}$ is termed the ***modulus of rigidity*** of the material.

Referring to Fig. 6·3, if s is the area of the upper face of the block the shearing stress is $\frac{F}{s}$, and since the strain is $\tan\theta$, or θ (expressed in circular measure), if the angle of shear $\widehat{CBX}$ is small, the modulus of rigidity is $\frac{F}{s} \div \theta$. The above method of determining k for

shearing stresses is only applicable to india-rubber, for the angle of shear is usually so small that it cannot be measured directly. Other methods are therefore employed, but they are beyond the scope of the present work.

Experiment.—Obtain a rectangular block of indiarubber 20 cm. long, 4 cm. wide, and 5 cm. thick, and cement one of the 20 cm. × 4 cm. faces to a vertical wall, the long edge being vertical. Cement a thin metal plate to the face opposite that cemented to the wall and suspend various loads by means of a hook attached to the plate. Measure with the aid of a travelling microscope the descent of the plate for each load and calculate the mean descent for unit change in load. Calculate the rigidity, n, of indiarubber as indicated in the following example.

Example.—The mean extension for a change in load of 1 kgm. was 0·040 mm. for the above block. Find the modulus of rigidity n.

$$\therefore \text{Angle of shear} = \frac{0{\cdot}004}{5} \text{ radian.}$$

$$\text{Change in stress} = \frac{1000 \times 981}{20 \times 4} \text{ dyne. cm.}^{-2}$$

$$\therefore \text{Modulus of rigidity} = \left(\frac{1000 \times 981}{20 \times 4} \times \frac{5}{0{\cdot}004}\right) \text{ dyne. cm.}^{-2}$$

$$= 1{\cdot}53 \times 10^7 \text{ dyne. cm.}^{-2}$$

Young's Modulus.—Suppose that a wire of length L and radius r is stretched by a load of mass m, the wire being free to contract in a direction perpendicular to the stretching force. The stress in the wire is $\frac{mg}{\pi r^2}$, since the wire is subject to stretching forces equal in magnitude to the weight of the load. If l is the increase in length the strain is $\frac{l}{\text{L}}$, and the modulus of elasticity, denoted by Y in this instance, is given by

$$\text{Y} = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{mg}{\pi r^2} \div \frac{l}{\text{L}} = \frac{mg\text{L}}{\pi r^2 l}.$$

Experimental Determination of Young's Modulus.—Two identical wires of the material under investigation are suspended from a beam. The method of attaching the wires to this support is important, for if either wire slips the elongation will not be due to the load alone. One method is to pass one long piece of wire between two brass plates which are afterwards screwed together with the aid of two or more bolts and nuts—cf. Fig. 6·5. This wire is arranged so that the lengths of the two free portions are approximately equal. A scale graduated in mm. is screwed to the left-hand wire whilst a second wire carries a vernier and a scale-pan. In this way, since the wires are identical, any temperature change will affect each wire to the same extent, so that no differential

expansion due to temperature variations will be noticed. The wire carrying the mm. scale is slightly stretched by suspending from it a convenient load. The pan attached to the second wire is usually sufficient to keep it straight when it is otherwise unloaded. The initial reading of the scales having been noted, the wire carrying the vernier is suitably loaded and the scale and vernier reading observed. To determine the ratio $\frac{\text{stress}}{\text{strain}}$ it is advisable to increase the load by 500 gm. at a time and observe the scale reading after each increment has been made: a graph showing the relationship between the load [ordinate] and the elongation [abscissa] is then constructed. Since the elongation is proportional to the load if the elastic limit has not been exceeded, this graph should be a straight line. With a piece of black cotton as a guide, the best straight line should be drawn through the points on the diagram and the slope calculated [cf. p. 14]. If θ is the slope of this line, we have

$$Y = \frac{mg}{\pi r^2} \cdot \frac{L}{l} = \frac{Lg}{\pi r^2} \cdot \theta.$$

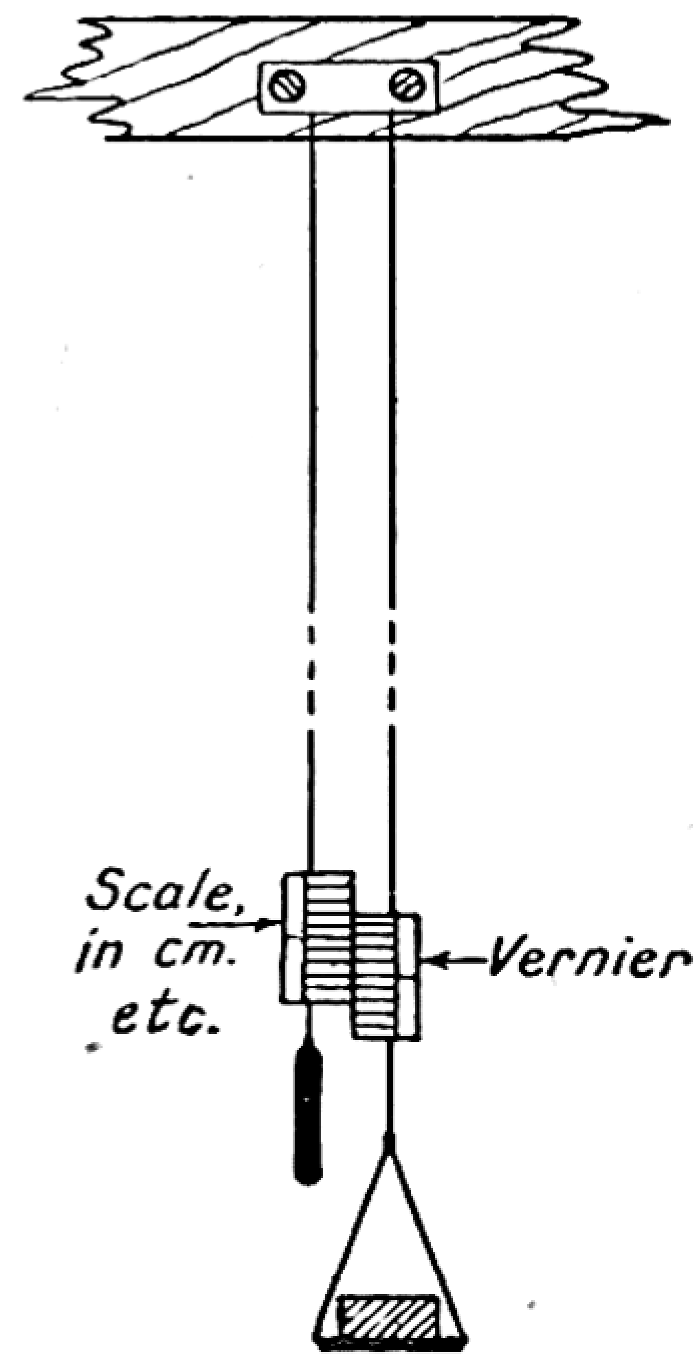


FIG. 6·5.—Apparatus for determining Y.M. for Metals in the form of Wires.

Young's modulus can therefore be calculated if, in addition to the above observations, the length and mean radius of the wire are known. The mean radius is determined with the aid of a micrometer screw gauge [cf. p. 7]. To test whether or not the elastic limit has been exceeded observations should also be made as the load is removed; corresponding observations will be in agreement, the wire returning to its original length, if the elastic limit has not been passed and the mean value of the extension for each load should be used in constructing the above graph.

Searle's Apparatus for Determining Young's Modulus for the Material of a Long Wire.—Two wires of the same material are hung from the same rigid support, their lengths being about 2 metres. Each carries at its lower end a brass rectangular frame from the lower sides of which suitable loads may be supported. In Fig. 6·6, A and B are the wires while C and D represent an end-on view of these frames. E is one of two bars freely hinged to the frames so that one frame may be displaced relatively to the other. H is a metal strip, carrying a spirit level S, and freely moving about a fulcrum M at one end. At the other end it rests upon the point N of a vertical screw R, operated by the divided head T. The pitch of the screw is 0·5 mm. and the periphery of T is divided into 50 equal divisions. When the head T is rotated through one division its point moves 0·01 mm.

A load of 1 kgm. is applied to each wire so that they shall be straight and the reading of the screw observed when one end of the air bubble is at the centre of the level. [This permits the position of the bubble to be adjusted more precisely than if it is attempted to adjust the bubble to a central position.] The load on one wire is then increased by 1 kgm. so that the wire is stretched and the air bubble displaced. By rotating the screw this bubble may be brought back to its standard position. The amount by which the point of the screw is moved is equal to the extension of the wire. The load is then increased in stages up to a maximum, removed 1 kgm. at a time, and readings of the screw taken for each load. A graphical or other method is then used to determine the mean extension for an increase in load of 1 kgm. and a value for Young's modulus for the material of the wire calculated as in the previous experiment.

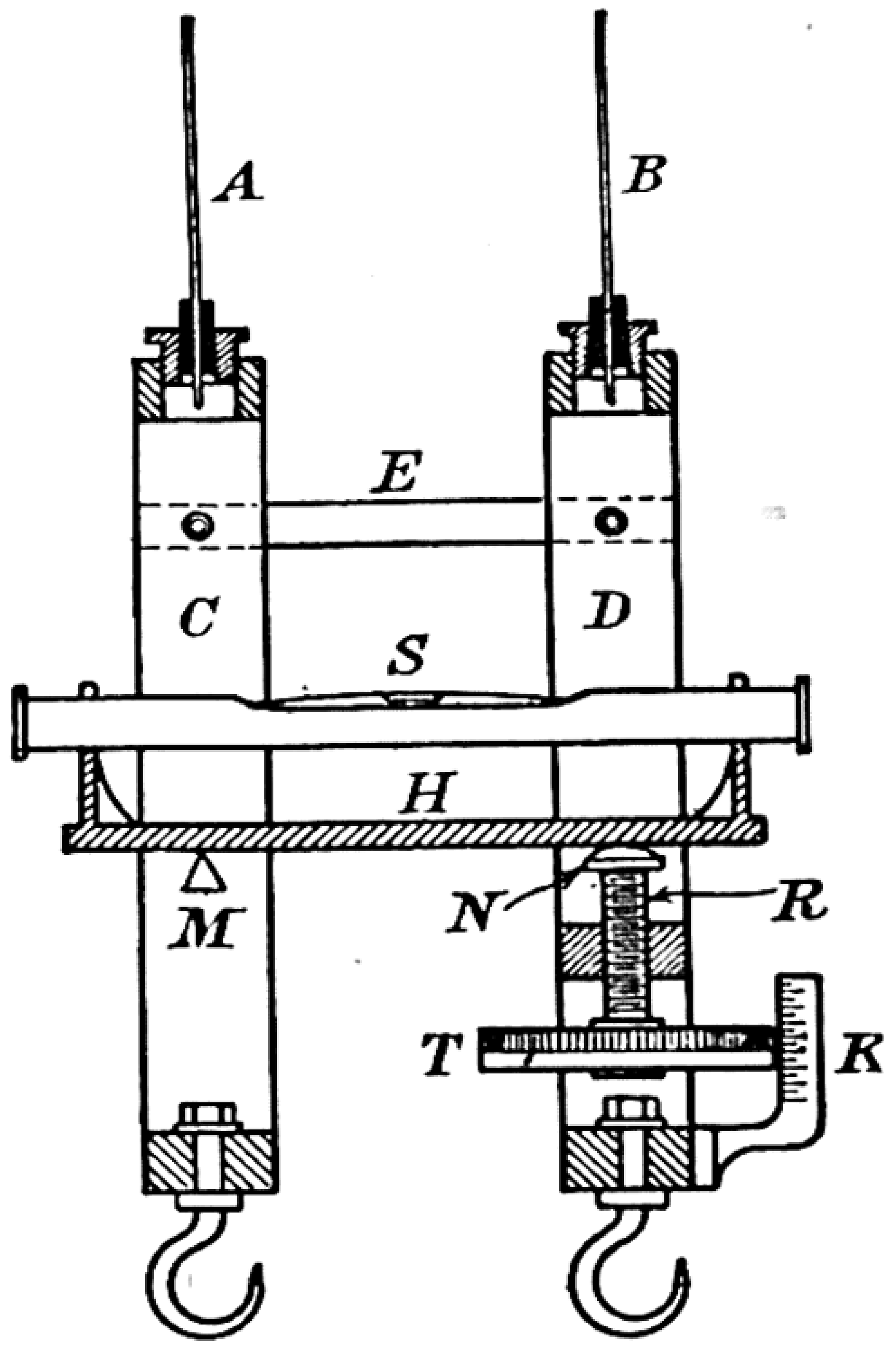


FIG. 6·6.—Searle's Apparatus for investigating the Stretching of Wires.

Poisson's Ratio.—When a wire is subjected to the action of stretching forces only, in addition to the elongation which occurs, there is a contraction in all directions perpendicular to the length of the wire. The change in diameter relative to the original diameter is termed the ***lateral contraction strain.*** The ratio of the lateral strain to the elongation strain is known as ***Poisson's ratio*** (σ). For indiarubber available in the form of a long solid tyre, this ratio may be determined from observations on the change in diameter and the change in length. For other substances Poisson's ratio is calculated from the other elastic constants for each particular substance. The necessary formulae are too difficult to prove here.

Volume Elasticity or Bulk Modulus.—We have seen that if a body is subjected to a uniform pressure its volume diminishes. If p is the increase in pressure necessary to cause a volume V of a material to diminish by an amount v, the stress is p, for a pressure is defined as a force per unit area [cf. p. 71], while the strain is $\frac{v}{V}$. The modulus of elasticity, which, in such an instance, is termed the ***volume elasticity*** or ***bulk modulus,*** is therefore $p \div \frac{v}{V}$, i.e. $\frac{pV}{v}$,

and is denoted[1] by β. The reciprocal of the bulk modulus is termed the ***compressibility*** of the substance, and is denoted by κ, so that $\kappa = \frac{1}{\beta}$.

The Compressibility of Liquids.—In an elementary account of hydrostatics it is always assumed that liquids are incompressible and, in fact, enormous pressures are required to alter by a small amount the volume of unit volume of all liquids, i.e. their compressibilities are always small. The fact that water was not a liquid with zero compressibility was first established by CANTON in 1762. BACON, at an earlier date, had subjected water, completely filling a lead sphere, to pressures greater than atmospheric, but the experiments were frustrated by the fact that the sphere always sprang a leak, or else the water escaped through the walls of the vessel which were porous. Canton used a glass vessel shaped like a thermometer and containing mercury, but with an open capillary tube. The level of the mercury in the stem of the instrument at a definite temperature was noted. The vessel was then heated until the mercury just filled it: the open end of the capillary was sealed and the instrument allowed to cool to its former temperature. It was found that the mercury stood at a higher level in the capillary than formerly. To account for this it might be assumed:—

(i) that the mercury had previously been compressed by the external air, or
(ii) that the vessel was reduced in size when the pressure inside was less than atmospheric.

The experiment was then repeated with water in the same vessel: the change in level of the water was greater than in the case of mercury. It was therefore established that water was a compressible substance.

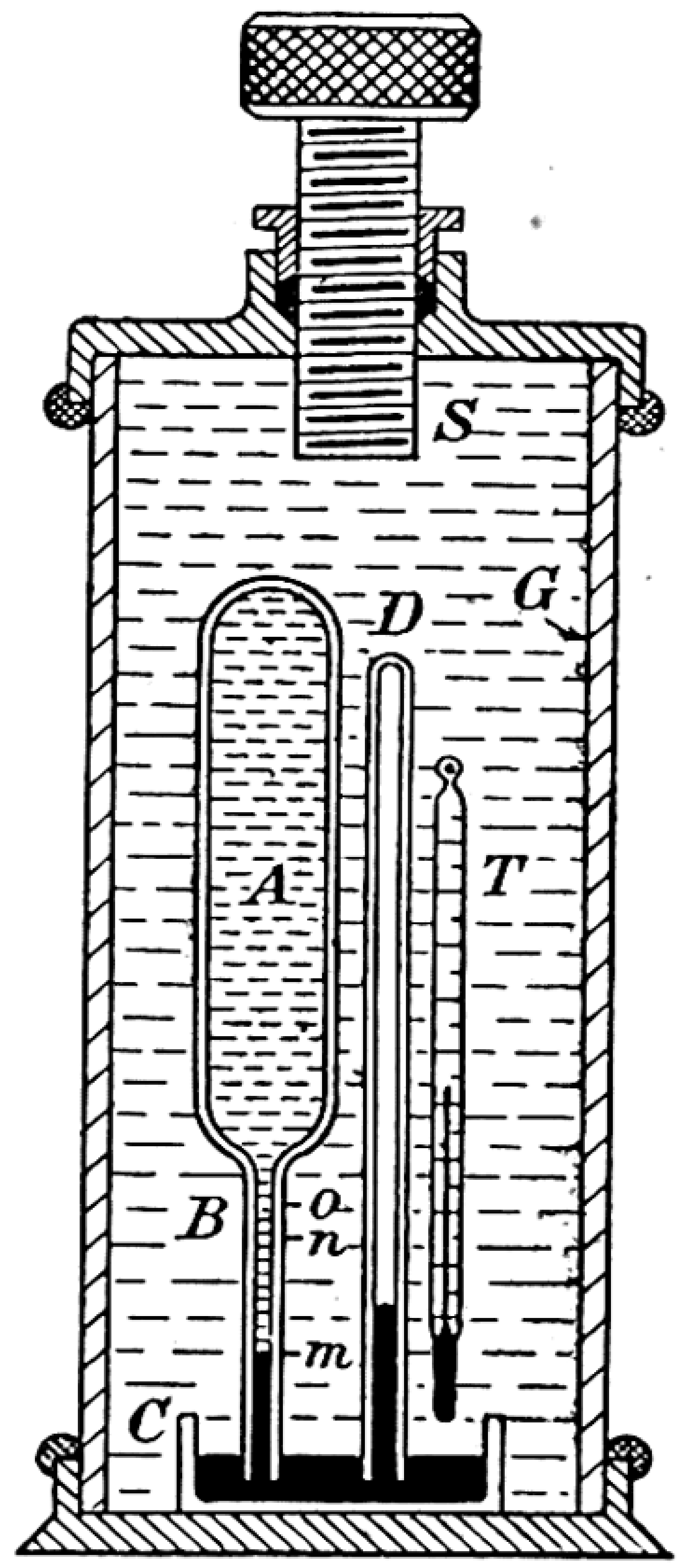


FIG. 6·7.—The Compressibility of Liquids [after Oersted].

Experimental determinations of the compressibilities of liquids are beset with many difficulties, but the underlying principles are shown by the following experiment due to OERSTED (1822). One form of his apparatus—an example of a class of instruments known as *piezometers*—is shown in Fig. 6·7. The liquid under investi-

[1] In acoustics E is used instead of β.

gation was contained in a cylindrical glass vessel, A, provided with a capillary tube, B, dipping below the surface of mercury contained in a trough, C. D was a narrow uniform glass tube, closed at the top, whose lower end dipped into the same dish of mercury. It contained air. The whole was placed in a wide glass tube, G, completely filled with water. Packing glands prevented the escape of water from between the ends of G and the metal discs closing its ends. By rotating the screw H so that it moved downwards very large pressures were exerted inside G, and these were transmitted to the liquid in A. The change in pressure inside the apparatus was deduced from the change in volume of the air in D. ***Theory:*** Let V be the volume of A up to the zero mark 0 on its stem. Let v be the volume per division on the stem B. Suppose that when the pressure was p_1, the mercury in B stood at m: when it was p_2 at n. Then $(m - n)v$ is ***apparently*** the reduction in volume of a volume $(V + mv)$ of liquid when the pressure changes by an amount $(p_2 - p_1)$. The ***apparent compressibility*** is therefore given by

$$\kappa = \frac{\text{apparent diminution in volume}}{(\text{original volume}) \times (\text{change in pressure})} = \frac{(m - n)v}{(V + mv)(p_2 - p_1)}.$$

The more exact theory, due to Lamé, shows that from the rise of mercury in B, together with other relevant data, the difference between the compressibilities of the liquid and glass may be deduced.

Regnault, in 1847, was the first person to obtain accurate values for the compressibilities of several common liquids, including mercury. In interpreting his observations he made use of the theoretical investigations of Lamé.

Experiment.—The smallness of the compressibility of water is shown by the following experiment. Water completely fills a metal box having no lid. When a small bullet is fired into one side of this box the volume of water is diminished by an amount practically equal to that of the bullet before the water has had time to rise. Consequently enormous pressures are exerted on the sides of the box and this bursts.

The Volume Elasticity or Bulk Modulus of an Ideal Gas at Constant Temperature.—Let P and V be the pressure and volume of a given mass of an ideal gas at constant temperature. Let the pressure become $P + p$, the corresponding volume being $V - v$. Then the increase in stress is p, while the corresponding strain is $\frac{v}{V}$, so that, by definition, the bulk modulus is $p \div \left(\frac{v}{V}\right)$. Since the gas is an ideal one and therefore obeys Boyle's law

$$(P + p)\ (V - v) = PV,$$

or

$$pV - vP - pv = 0.$$

If p and v are small compared with P and V their product may be neglected, so that

$$pV - vP = 0, \text{ or } P = \frac{pV}{v} = \beta.$$

The volume elasticity of an ideal gas at constant temperature is therefore equal to the pressure to which it is subjected. [N.B.—The pressure must be expressed in absolute or in gravitational units.]

Alternative Proof. Let P and V become $P + \delta P$ and $V + \delta V$ respectively. The increase in stress is δP, while the strain is $-\frac{\delta V}{V}$.

$$\therefore \text{ by definition, } \beta = \lim_{\delta P \to 0}\left[-\frac{\delta P}{\left(\frac{\delta V}{V}\right)}\right] = \lim_{\delta P \to 0}\left[-V\frac{\delta P}{\delta V}\right].$$

Now by Boyle's law, $(P + \delta P)(V + \delta V) = PV$.

$$\therefore V\delta P + P\delta V = 0,$$

since the product $\delta P . \delta V$ may be neglected. Hence

$$\lim_{\delta P \to 0}\left[-V \cdot \frac{\delta P}{\delta V}\right] = P, \text{ i.e. } \beta = P.$$

Energy due to Strain.—In order to deform a body work must be done by the applied forces. The energy thus spent is stored in the body which is then said to possess ***strain energy***. This energy is lost when the stress is removed, appearing as heat, i.e. the body is temporarily at a temperature above that of its surroundings. The whole of the work done in deforming the body is only completely regained if its elastic limit has not been passed, for in this latter instance a permanent set is produced in the body and the energy necessary to do this is not regained when the stress is removed.

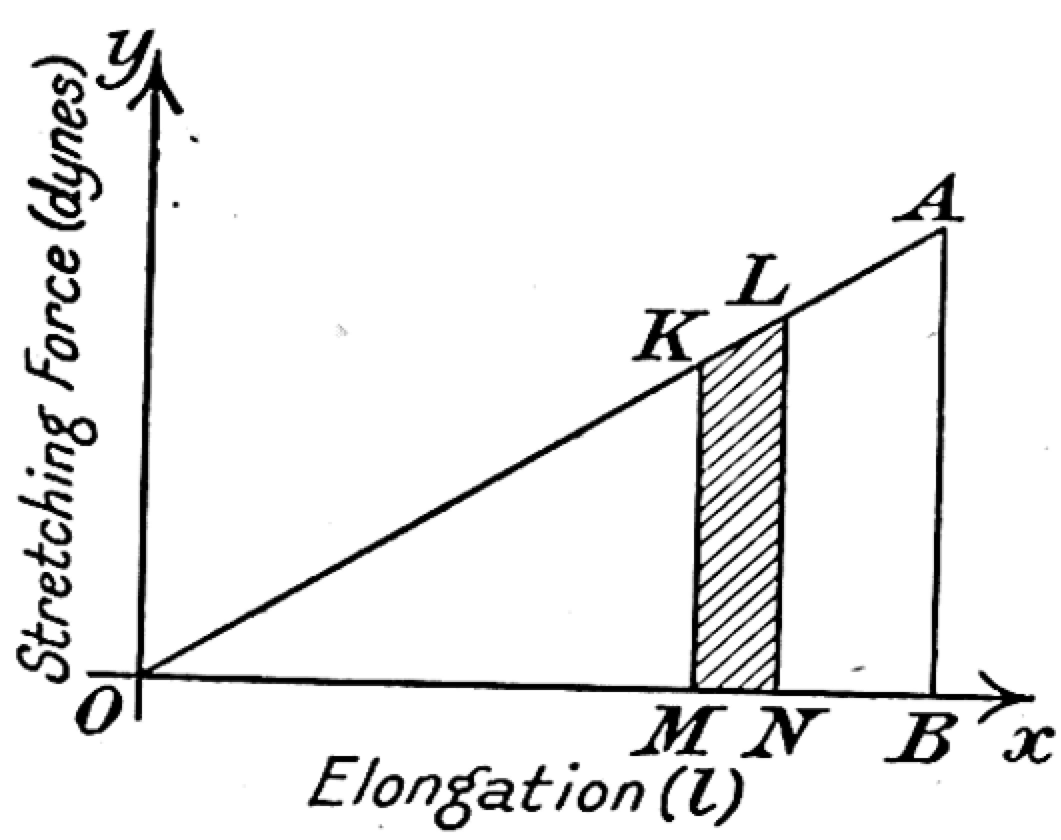


FIG. 6·8.—Energy due to Strain.

For a wire in which the stress does not exceed its elastic limit the amount of work done in stretching it may be calculated as follows:—If a point A, Fig. 6·8, represents the state of the wire when the stretching force is F and the elongation l, the work done in producing this condition is represented by the area of the triangle OAB. To prove this, consider two points K and L on OA, and draw KM and LN perpendicular to the x-axis. If the points K and L are very close together then during the deformation MN

the stretching force may be considered constant and equal to that represented by KM, so that the work done is represented by KM $\times$ MN, i.e. it is represented by the area of the rectangle KLNM. Similarly, every such small rectangle into which the triangle OAB may be divided represents a quantity of work done. The total work done in deforming the wire is represented by the sum of all these rectangles, i.e. the area OAB. The work represented by this area is $\frac{1}{2}F \times l$. If L is the length of the wire and r its radius, the strain energy per unit volume is

$$\tfrac{1}{2}Fl \div \pi r^2 L = \tfrac{1}{2} \cdot \frac{F}{\pi r^2} \cdot \frac{l}{L}.$$

This equation means that the strain energy per unit volume is one half the product of the stress and the strain.

The Behaviour of Solids when the Applied Stress exceeds their Elastic Limits.—(*a*) *Brittle Materials in Tension.* Cast iron, hardened iron, Portland cement, stone and brick are examples of a brittle substance. Fig. 6·9 (*a*) shows the relation between the strain and stress for such a substance. OA is linear, so that A is the elastic limit, but beyond A the graph is curved. The point B represents the stage when the substance breaks.

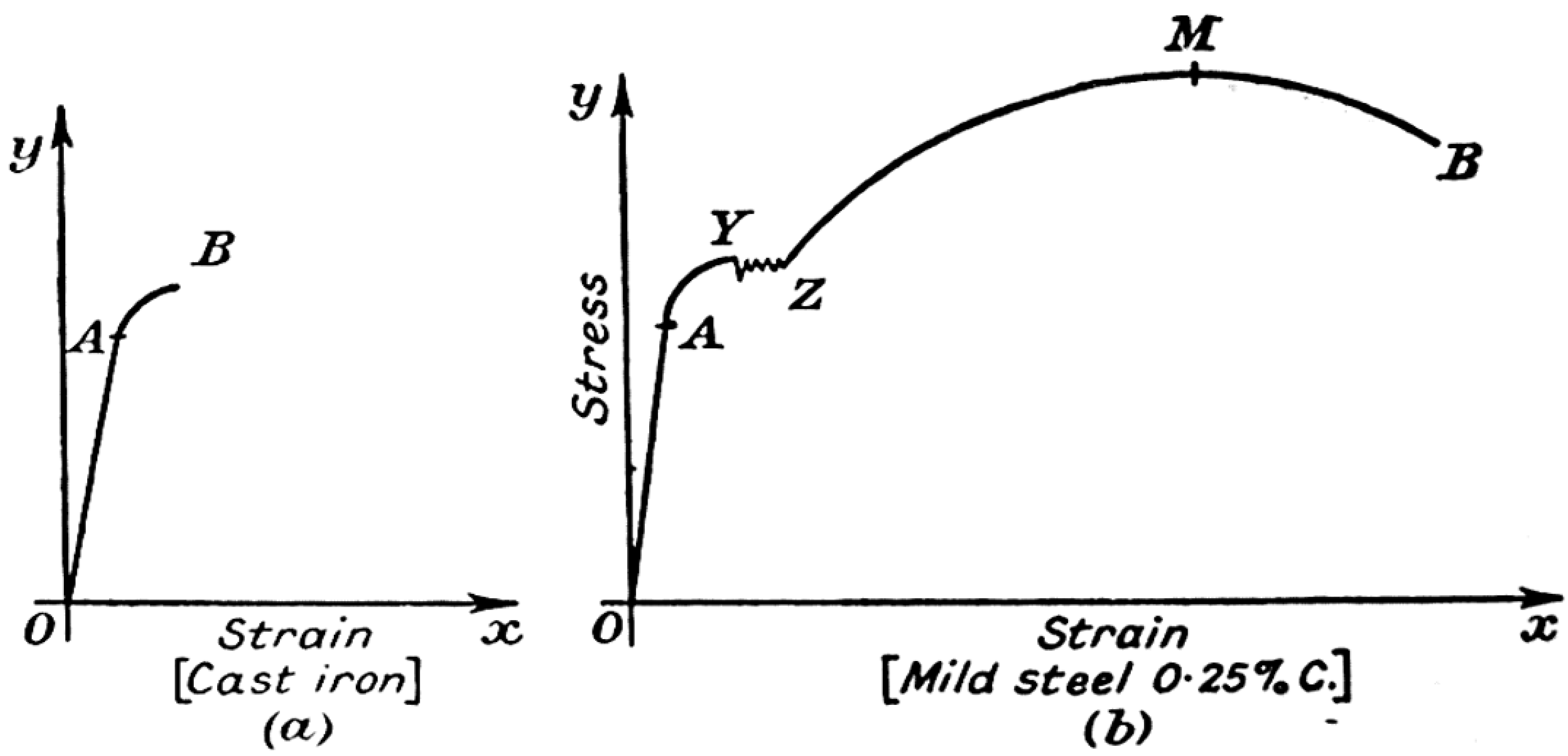


FIG. 6·9.—Behaviour of Solids when the Applied Stress exceeds their Elastic Limits.

(*b*) *Ductile Materials in Tension.* The stress-strain diagram for a 0·25 per cent. carbon steel, obtained autographically, is shown in Fig. 6·9 (*b*). A is the elastic limit and beyond this point the graph curves until the point Y is reached. Then comes the portion YZ of the curve, representing the stage during which there is a large increase in strain with practically no increase in stress: on a self-recording sensitive extensometer the portion YZ appears as an irregular wavy line, the stress corresponding to Z being less than

that at Y. Y is called the ***yield point,*** the corresponding stress being the ***yield-stress.*** In the stage AY the stretch is partly elastic and partly due to plastic flow in minute conglomerates of particles distributed throughout the material under test. Beyond Z the elongation becomes plastic: during the elastic stage the stretch is caused by simple tension but in the plastic stage strain becomes predominant so that the stretch is mainly due to total shear taking place throughout all parts of the specimen. As the stress is increased the stretch proceeds steadily until the bar is about to fracture. Then a stage with marked instability sets in and the piece becomes considerably thinner at one point, i.e. the specimen exhibits a local contraction and a marked roughening of the hitherto smooth machined surface of the material appears. The specimen exhibits a phenomenon known as 'necking.' Immediately this occurs the stress decreases automatically and the portion MB of the curve is obtained: the break finally occurs at B. The stress corresponding to M is called the ***ultimate strength*** or ***tensile strength*** of the material under test. Steel (0·2 per cent. carbon) has a tensile strength of 30 ton-wt. in.$^{-2}$, while among timbers, British Oak, 'that synonym for strength and durability,' with a tensile strength of about 7 ton-wt. in.$^{-2}$, stands supreme if certain foreign woods are excluded. Unfortunately, it contains acids which corrode iron and steel. It is for this reason that copper rivets are used in the construction of wooden ships.

Elastic Fatigue.—When a metal has been subjected to repeated alternations of stress it becomes 'fatigued,' i.e. its strength diminishes, which means that for a given stress the amount of strain increases. If the alternations are continued for a sufficiently long time the metal may ultimately develop a fracture.

In the manufacture of copper tubes of elliptical section the tubes are first drawn with a circular section. If the final operation of making the bore elliptical is carried out at once it is successful, but if the tube is allowed to remain overnight the process cannot be completed in the morning.

EXAMPLES VI

1.—Define the terms: *tensile stress, tensile strain, Young's modulus bulk modulus, compressibility.* Derive an expression for the bulk modulus of an ideal gas. Two uniform wires of the same material are such that the linear dimensions of one are double those of the other. If equal loads are suspended from the above wires calculate the ratio of the extensions produced.

2.—Derive an expression for the force inwards due to a rope under tension passing round a smooth curve. Calculate the limiting pressure inside a cylindrical boiler of 3 ft. radius, the sides being $\frac{1}{8}$ of an inch thick and made of a material which can stand a limiting pressure of 40 ton.-wt. in.$^{-2}$.

3.—How would you proceed to determine Young's modulus for a substance in the form of a uniform wire? If Y.M. for steel is 2×10^{12} dyne. cm.$^{-2}$, what mass must be suspended from a steel wire 2 metres long and 1 mm. diameter to stretch it by 1 mm.?

4.—A solid has a volume of 3·5 litres when the external pressure is 1 atmosphere. If the bulk modulus of its material is 10^{12} dyne. cm.$^{-2}$, calculate the change in volume when the body is subjected to a pressure of 25 atmospheres.

5.—Explain Hooke's law and describe how you would proceed to verify it for the extension of a vertical wire under load. A copper wire, 2 metres long and 3 mm.^{2} cross-sectional area, is suspended vertically and a load of 5 kilograms attached to its lower end. Calculate the work done in stretching the wire if Young's modulus for copper is $1{\cdot}2 \times 10^{12}$ dyne. cm.$^{-2}$

6.—Define *Young's modulus* and the *modulus of bulk elasticity*. Calculate the value of the latter modulus for a substance of which 1 cubic decimetre is reduced in volume by 0·01 cm.^{3} by an increase of pressure of 20 atmospheres.

7.—Explain what is meant by the statement: 'Young's modulus for steel is 2×10^{12} dyne. cm.$^{-2}$' Calculate the mass of the load which must be suspended from a steel wire 1 mm. in diameter to produce an elongation equal to 0·2 per cent. of its original length.

8.—How would you compare experimentally the value of Young's modulus for copper with the value for the modulus of brass, being given wires of the same standard gauge?

9.—Calculate the modulus of bulk elasticity for a substance of which 1 cubic decimetre is reduced in volume by 0·004 cm.^{3} when subjected to an increase of pressure of 16 atmospheres.

10.—Calculate the density of water at the bottom of a lake 150 metres deep assuming that the compressibility of water is $\frac{1}{21,000}$ atmos.$^{-1}$

11.—Given that Young's modulus for steel is 2×10^{12} dyne. cm.$^{-2}$ calculate its value in pounds weight per square inch.

12.—A spiral spring of negligible mass is hung vertically and is such that a load of 6·5 gm.-wt. produces an extension of 10 cm. If the spring carrying a load of 508 gm.-wt. is pulled downward, show that the load will execute a S.H.M. when the spring is released, and determine its period.

13.—An elastic string of natural length $2a$ can just support a certain weight when it is stretched until its whole length is $3a$. One end of the string is now attached to a point in a smooth horizontal table, and the same weight is attached to the other end and can move on the table. Prove that if the weight is pulled out to any distance and then let go, the string will become slack again after a time $\frac{\pi}{2}\sqrt{\frac{a}{g}}$.—(L.I.)

14.—A mass of metal of volume 500 cm.^{3} hangs on the end of a wire whose upper end is rigidly fixed. The diameter of the wire is uniform and equal to 0·4 mm. and its Young's modulus 7×10^{11} dyne. cm.$^{-2}$ When the metal is completely immersed in water, the length of the wire is observed to change by 1 mm. Find the length of the wire if the acceleration due to gravity is 980 cm. sec.$^{-2}$—(N.H.S.C. '29).

ANSWERS TO THE EXAMPLES

I. (1) 0·82, 0·73, 0·68, 1·07, 57° 18′. (4) 3·14 ft. (5) 24·9 cm.

II. (1) 41·7 ft. (2) 4° 47′. (3) 0·92 ft. sec.$^{-2}$, 6·0 sec. (4) 8 cm. sec.$^{-2}$ (5) 4·79 sec., 367 ft. (6) 2·7 sec. (7) 48 ml. hr.$^{-1}$ (8) 67·9 ft. sec.$^{-1}$, 102 ft. (9) 20 cm., 40 cm. (10) 110 ft. lb.-wt. (11) 0·447, 1. (12) $20\pi/3$, $40\pi^2/9$. (13) 1·01 ton.-wt. (14) $2\pi/7$ sec. (15) $4{\cdot}3 \times 10^6$ erg.
(16) $10\ m\sqrt{lg(2-\sqrt{3})}$ gm. cm. sec.$^{-1}$, $50\ mlg(2-\sqrt{3})$ erg.

III. (1) 139 ft., 324 lb. ft. sec.$^{-1}$ (2) 11·7 lb.-wt. 3·34 ft. sec.$^{-2}$ (4) 0·29 cm. from the centre. (5) 42·2 lb.-wt. (6) 28 lb.-wt., 1 : 3. (8) 20·67 gm. (9) 1·55 ft. from the fulcrum. (10) 78·7 gm. (11) 62° 55′. (13) Any point in a vertical line 0·5 in. from the median through C, and on the side nearer to B. (17) 86 ft. (18) 16·8 ft. (19) 26′.

IV. (1) 20·4 gm. (2) 40·6 cm. (3) 160·3 atmos., 152·1 ton.-wt. ft.$^{-2}$ (4) 211 gm.-wt. cm.$^{-2}$ (5) 0·73 ft. (6) 6·88 in. (10) 4·95 cm.^{3} (11) 29·2 in. of mercury. (12) 261·1 gm. (13) 0·0779 cm., 1·297 gm. cm.$^{-3}$ (14) 13·7 cm. of water. (15) 12·5 cm.^{3} (16) 3,809 lb. yd.$^{-3}$ (18) 0·604 : 1 by volume. (19) 0·0169 gm. too heavy. (20) 7·57 cm. (21) 1·014. (22) 75 cm. (23) 57 cm. (25) 29·54 in. (26) 29·5 in.

V. (3) 30·5 dyne. cm.$^{-1}$ (4) 2·7 gm. cm.$^{-1}$ sec.$^{-1}$ (8) 30·7 dyne. cm.$^{-1}$ (13) $1{\cdot}25 \times 10^5$ dyne. cm.$^{-2}$ (15) 2·50 cm. (16) 1·3 cm. (17) 15·1 cm.

VI. (1) 2. (2) 311 lb.-wt. in.$^{-2}$ (3) 8 kgm. (4) 0·085 cm.^{3} (5) $6{\cdot}68 \times 10^4$ erg. (6) $2{\cdot}03 \times 10^{12}$ dyne. cm.$^{-2}$ (7) 32 kgm. (9) $4{\cdot}05 \times 10^{12}$ dyne. cm.$^{-2}$ (10) 1·0007 gm. cm.$^{-3}$ (11) $1{\cdot}29 \times 10^4$ lb.-wt. in.$^{-2}$ (12) 1·76 sec. (14) 179·5 cm.

TRIGONOMETRICAL RATIOS

Angle.									
Degrees.	Radians.	Chord.	Sine.	Tangent.	Co-tangent.	Cosine.			
0°	0	0	0	0	∞	1	1·414	1·5708	90°
1	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	89
2	·0349	·035	·0349	·0349	28·6363	·9994	1·389	1·5359	88
3	·0524	·052	·0523	·0524	19·0811	·9986	1·377	1·5184	87
4	·0698	·070	·0698	·0699	14·3007	·9976	1·364	1·5010	86
5	·0873	·087	·0872	·0875	11·4301	·9962	1·351	1·4835	85
6	·1047	·105	·1045	·1051	9·5144	·9945	1·338	1·4661	84
7	·1222	·122	·1219	·1228	8·1443	·9925	1·325	1·4486	83
8	·1396	·140	·1392	·1405	7·1154	·9903	1·312	1·4312	82
9	·1571	·157	·1564	·1584	6·3138	·9877	1·299	1·4137	81
10	·1745	·174	·1736	·1763	5·6713	·9848	1·286	1·3963	80
11	·1920	·192	·1908	·1944	5·1446	·9816	1·272	1·3788	79
12	·2094	·209	·2079	·2126	4·7046	·9781	1·259	1·3614	78
13	·2269	·226	·2250	·2309	4·3315	·9744	1·245	1·3439	77
14	·2443	·244	·2419	·2493	4·0108	·9703	1·231	1·3265	76
15	·2618	·261	·2588	·2679	3·7321	·9659	1·218	1·3090	75
16	·2793	·278	·2756	·2867	3·4874	·9613	1·204	1·2915	74
17	·2967	·296	·2924	·3057	3·2709	·9563	1·190	1·2741	73
18	·3142	·313	·3090	·3249	3·0777	·9511	1·176	1·2566	72
19	·3316	·330	·3256	·3443	2·9042	·9455	1·161	1·2392	71
20	·3491	·347	·3420	·3640	2·7475	·9397	1·147	1·2217	70
21	·3665	·364	·3584	·3839	2·6051	·9336	1·133	1·2043	69
22	·3840	·382	·3746	·4040	2·4751	·9272	1·118	1·1868	68
23	·4014	·399	·3907	·4245	2·3559	·9205	1·104	1·1694	67
24	·4189	·416	·4067	·4452	2·2460	·9135	1·089	1·1519	66
25	·4363	·433	·4226	·4663	2·1445	·9063	1·075	1·1345	65
26	·4538	·450	·4384	·4877	2·0503	·8988	1·060	1·1170	64
27	·4712	·467	·4540	·5095	1·9626	·8910	1·045	1·0996	63
28	·4887	·484	·4695	·5317	1·8807	·8829	1·030	1·0821	62
29	·5061	·501	·4848	·5543	1·8040	·8746	1·015	1·0647	61
30	·5236	·518	·5000	·5774	1·7321	·8660	1·000	1·0472	60
31	·5411	·534	·5150	·6009	1·6643	·8572	·985	1·0297	59
32	·5585	·551	·5299	·6249	1·6003	·8480	·970	1·0123	58
33	·5760	·568	·5446	·6494	1·5399	·8387	·954	·9948	57
34	·5934	·585	·5592	·6745	1·4826	·8290	·939	·9774	56
35	·6109	·601	·5736	·7002	1·4281	·8192	·923	·9599	55
36	·6283	·618	·5878	·7265	1·3764	·8090	·908	·9425	54
37	·6458	·635	·6018	·7536	1·3270	·7986	·892	·9250	53
38	·6632	·651	·6157	·7813	1·2799	·7880	·877	·9076	52
39	·6807	·668	·6293	·8098	1·2349	·7771	·861	·8901	51
40	·6981	·684	·6428	·8391	1·1918	·7660	·845	·8727	50
41	·7156	·700	·6561	·8693	1·1504	·7547	·829	·8552	49
42	·7330	·717	·6691	·9004	1·1106	·7431	·813	·8378	48
43	·7505	·733	·6820	·9325	1·0724	·7314	·797	·8203	47
44	·7679	·749	·6947	·9657	1·0355	·7193	·781	·8029	46
45°	·7854	·765	·7071	1·0000	1·0000	·7071	·765	·7854	45°
			Cosine.	Co-tangent.	Tangent.	Sine.	Chord.	Radians.	Degrees.
								Angle.	

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934	0969					3	7	11	14	18	21	25	28	32
							1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553							3	6	9	12	15	18	21	24	28
					1584	1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875	1903					3	6	9	11	14	17	20	23	26
							1931	1959	1987	2014	3	5	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405	2430					3	5	8	10	13	15	18	20	23
							2455	2480	2504	2529	2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5.
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

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